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School Choice with Preference Rank Classes*

Nickesha Ayoade† Szilvia Pápai‡

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Abstract

We introduce and study a large family of rules for many-to-one matching problems, the Preference Rank Partitioned (PRP) rules. PRP rules are student-proposing Deferred Acceptance rules, where the schools select among applicants in each round taking into account both the students' preferences and the schools' priorities. In a PRP rule each school first seeks to select students based on priority rank classes, and subsequently based on preference rank classes. PRP rules include many well-known matching rules, such as the standard Deferred Acceptance rule, the Boston rule, the Chinese Application-Rejection rules of Chen and Kesten (2017), the Taiwan Deduction rules of Dur et al. (2018), and the French Priority rules of Bonkoungou (2019), in addition to matching rules that have not been studied yet. We analyze the stability, efficiency and incentive properties of PRP matching rules in this unified framework.

Keywords: matching; school choice; Deferred Acceptance; Boston rule; stability

1 Introduction

We study a large family of matching rules, which includes many already well-known rules, for many-to-one matching problems that assign heterogeneous indivisible objects

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to agents, where objects have strict priorities over agents and several agents may be assigned to the same object. Since this model is known as the school choice model due to Abdulkadiroğlu and Sönmez (2003), we call the objects schools and refer to the agents as students. However, the theoretical approach and results pertain to a broad range of applications, not just to school choice, such as centralized university admissions, refugee resettlement, and dormitory room assignments, among others. Balinski and Sönmez (1999) introduced this model first, which only differs from the college admissions model of Gale and Shapley (1962) in that school priorities are mandated by policies or by the law and thus school seats can be viewed as objects to be allocated, while in college admissions schools have preferences and are considered to be strategic agents. This has implications for the efficiency and incentive axioms used in the two models. In the school choice model, only the students' welfare and incentives are considered. Stability, on the other hand, translates into fairness in the school choice model, since the exogeneously given school priorities are taken into account from the students' perspective.

We call the large family of rules that we introduce and study Preference Rank Partitioned (PRP) rules, since these matching rules are student-proposing Deferred Acceptance (DA) rules (Gale and Shapley, 1962) in which schools use a choice function to select among applicants that rely not only on the school priorities, but also on a partitioning of student preferences. Choice-based DA mechanisms are studied and characterized by Kojima and Manea (2010) and Ehlers and Klaus (2016). Choice functions are employed in matching with diversity constraints (e.g., Ehlers et al., 2014), matching with distributional constraints (e.g., Kamada and Kojima, 2018), or more generally by numerous papers in matching with contracts. None of these papers consider choice functions which depend on preferences.

PRP rules are determined by a partition of each school's priority ordering of students and by a partition of each student's preference ranking over schools, which lead to priority and preference rank classes respectively. Students who are in a higher priority rank class are selected by the school's choice function first, followed by a comparison of the preference rank classes in which applying students place the school in question, in order to make further selections. If ties remain then the school-specific strict priorities over students are used for tie-breaking. Since the given priorities are assumed to be strict, a PRP matching rule specifies, as the first selection criterion, priority rank classes which lead to coarse priorities that are consistent with the given strict priorities. As the second selection criterion, a PRP rule bases the selection of students on their preference rank classes in the instances where some students applying to the school are in the same priority rank class and selecting all of them would result in exceeding the school's capacity. The strict priorities within the priority rank classes specified by the PRP rule are used only for tie-breaking, as a last resort, when neither the priority rank classes nor the preference rank classes can determine the selection of students by a school in a particular round of the iterated Deferred Acceptance procedure. Thus, these matching rules can be interpreted as rules that first coarsen the given strict priorities and then refine them using preference rank classes. An alternative interpretation is that they start from given coarse priorities and use the preferences rank classes to refine the priorities. In the latter interpretation the tie-breaking, when necessary, can be done randomly.

The set of PRP rules includes many well-known matching rules, such as the Deferred Acceptance (Gale and Shapley, 1962) and Boston (Immediate Acceptance) rules (Abdulkadiroğlu and Sönmez, 2003), as well as the family of Application-Rejection mechanisms of Chen and Kesten (2017), the French Priority mechanisms of Bonkoungou (2019), the Taiwan Deduction mechanisms of Dur et al. (2018) and the Secure Boston mechanism introduced by Dur et al. (2019), among others. We analyze the family of PRP matching rules, and also study a subfamily of PRP rules, the Equitable PRP rules, which treat students symmetrically. For matching rules in this subfamily the preference rank classes are homogeneous across students. All previously studied rules and families of rules that are PRP rules are Equitable PRP rules. We specify new PRP rules which are not Equitable PRP rules, namely, the class of Favored Students rules, which treat students in one of two ways: each student has either the coarsest or the finest preference rank partition. We also identify some further classes of PRP rules which are Equitable PRP rules and include the Deferred Acceptance and Boston rules, but are distinct from such families of rules already studied in the literature, such as the Application-Rejection rules, the Taiwan Deduction rules, and the generalized class of Secure Boston rules, proposed by Dur et al. (2019). One such class is what we call the Deferred Boston rules, which are PRP rules that have homogeneous priority rank partitions combined with the finest preference rank partitions. We also identify the class of Homogeneous PRP rules, characterized by having both homogeneous priority and preference partitions, which makes Homogeneous PRP rules a superset of almost all previously studied PRP rules (except for some French Priority rules such as First Preference First and Secure Boston rules) and a subset of Equitable PRP rules.

Several PRP rules have been used worldwide in school choice, university admissions, and hospital-intern matching. Apart from the widely used Deferred Acceptance rule, which was adopted (with some variations) by the school boards of several large US cities such as New York City, Boston, and New Orleans, and is being used extensively in hospital-intern matching in North-America, the Boston rule was in use in Boston for school choice until 2005 and is still a popular procedure for student placement.[1] The Boston rule is a special case of the priority matching mechanisms of Roth (1991), which were used in several UK cities starting in the 1960s for allocating hospital positions to graduating medical students and were subsequently abandoned, since these mechanisms have poor stability and incentive properties. Priority matching mechanisms rely both on the exact preference ranks of hospitals by students and the exact priority ranks of students by hospitals to determine the matching, and use a formula which orders the pairs of ranks starting with $(1, 1)$ to make matches. The only priority matching rule that is a PRP rule is the Boston rule. The First Preference First rule used in England for school choice was banned in 2007 (Pathak and Sönmez, 2013). The French Priority rules are employed in centralized university admissions in France (Bonkoungou, 2019). The Parallel Mechanisms (more generally, Application-Rejection rules) are in use in China (Chen and Kesten, 2017). The Taiwan Deduction rules of Dur et al. (2018) have been used for high school assignment in Taiwan since 2014. Hence, PRP rules and their properties are not just of theoretical interest, but also have practical relevance.

Our results pertain to the stability, efficiency, and incentive properties of PRP rules. We first demonstrate that PRP rules choose the optimal matching that is consistent with the specific school choice functions at each preference profile (Proposition 1). We also characterize the subclass of PRP rules which treat students symmetrically, the Equitable PRP rules, by applying a natural weak stability property that relies on preferences in addition to the priorities to justify assignments (Proposition 2). This characterization generalizes to the larger class of rules which are not necessarily optimal but share the choice-function-specific stability properties of PRP rules (Theorem 1). We also show that the only Pareto-efficient PRP rules (Pareto efficient for the students' side of the market) are what we call the Near-Boston rules, and this result also holds

[1] See Pathak (2017) for the literature on the practical aspects of school choice design.

for the larger set of rules mentioned above, which form a superset of PRP rules (Theorem 2). Surprisingly, the set of Near-Boston rules includes some PRP rules other than the Boston rule, which itself is well-known to be efficient (Abdulkadiroğlu and Sönmez, 2003; Kojima and Ünver, 2014). However, the class of Pareto-efficient PRP rules is still quite restricted, since for these matching rules only one student's preference rank partition may differ from that of the Boston rule, which itself calls for the finest preference partition for each student. A serious issue with the efficiency of the Near-Boston rules is that these rules are not strategyproof (note that in this context for strategyproofness we only take into account the students' incentives, as the schools are not considered strategic agents). In fact, the Boston rule is well-known to be highly manipulable. Thus, since many students (or their parents, in the case of school choice) misrepresent their true preferences, the matching outcome is unlikely to be Pareto-efficient, and in fact the welfare loss can be significant (Ergin and Sönmez, 2006; Pathak and Sönmez, 2008). These concerns also carry over to the other Near-Boston rules.

We also prove that the only strategyproof PRP rule is the Deferred Acceptance rule (Theorem 3). Hirata and Kasuya (2017) study stability and strategyproofness in a general matching market with contracts, and find that the number of stable and strategy-proof rules is at most one. They also show that if the student-optimal stable rule exists then it is the only candidate for a stable and strategyproof rule. These results, nonetheless, don't imply ours, since in their setup the choice function selects a feasible set of contracts from each set of contracts independently of the preferences. Alva and Manjunath (2019) also study related topics.

While PRP rules are not strategyproof except for the DA, they can be shown to be less manipulable than their non-optimal counterparts, using the criterion of Pathak and Sönmez (2013) for comparing manipulability. This follows from the general results of Pathak and Sönmez (2013) and Chen et al. (2016). Our main result on incentives is that students cannot manipulate PRP rules to obtain a school that was unattainable when reporting their preferences truthfully by placing this school in the same preference rank class or in a lower one when reporting untruthfully (Theorem 4). This theorem sheds new light on the incentive properties of well-known PRP rules and offers insight into their manipulability properties in general.

The closest papers to ours are Bonkoungou (2019), who studies an important subclass of PRP rules, the French Priority rules, in a somewhat different approach, and

Chen and Kesten (2017) and Dur et al. (2018), both of which study a class of PRP rules which is distinct from the French Priority rules and only have two members in common: the DA and the Boston rule. Bonkoungou (2019) has coarse priorities as primitives of his model, thus the French Priority rules, which always have the finest preference partition for students (as in the Boston rule), collapse to one single rule in his paper. He explores the incentive properties of the French Priority rule from both an ex-ante and an ex-post perspective. Bonkoungou also introduces a notion called strategic accessibility, which serves as a basis for further manipulability comparisons (see also Bonkoungou and Nesterov, 2019). Our analysis is from the ex-post perspective, and our results complement the ex-post perspective results of Bonkoungou (2019). Specifically, Bonkongou (2019) makes comparisons based on how fine the given coarse priorities are, while our main theorem on incentives pertains to the preference rank classes. Chen and Kesten (2017) study a subclass of PRP rules in which the preference rank classes are homogeneous but the priority rank classes are the coarsest. The Taiwan Deduction rules of Dur et al. (2018) translate into the same class of PRP rules in our framework as the Application-Rejection rules of Chen and Kesten (2017). We generalize the findings of Chen and Kesten (2017) about the extreme members of the class of Application-Rejection rules (Theorem 2 and Theorem 3). The results of Chen and Kesten (2017) and Dur et al. (2018) on manipulability comparisons of different matching rules are independent of our main result on incentives (Theorem 4).

Although we assume that there are exogenously determined strict priorities, PRP matching rules are also useful to study situations where only coarse (weak) priorities are given, as would be typical for public schools that cannot strictly order the students based on a few criteria only, or for countries which cannot distinguish among all refugee families but wish to prioritize refugees with certain skills and attributes. In Bonkoungou's (2019) approach the exogenously given priorities are coarse, following the practice of universities in France, and consequently the use of preference rankings is viewed as breaking the ties in the given coarse priorities. In cases like this, when priorities are weak, the DA and most other prominent matching rules that rely on priorities are not well-defined, and the ties need to be resolved. Abdulkadiroğlu et al. (2009) propose to use a single tie-breaking procedure in the DA, which is a tie-breaking lottery that applies to each school, and show this tie-breaking method to be superior to the multiple tie-breaking procedure, which means separate tie-breaking lotteries for schools. Erdil and Ergin (2008) introduce stable improvement cycles, which is essentially a different tie-breaking procedure at different preference profiles with the aim of improving student welfare while preserving stability with respect to the coarse priorities. Further interesting papers on tie-breaking are Ehlers and Erdil (2010) and Ehlers and Westkamp (2018).

2 Model

Let S be the set of n students and C the set of $m \geq 3$ schools. Each school c has capacity $q_c \geq 1$. In order to simplify the exposition, we assume that $m \geq 3$ and that there exist schools $a, b, c \in M$ such that $q_a + q_b + q_c < n$. The latter assumption ensures that there is scarcity for at least the three schools with the least capacity, so we don't need to take care of special cases where this minimum condition doesn't hold.

Each student $s \in S$ has a preference relation P_s, a strict ordering over $C \cup \{0\}$, where assigning 0 to student s represents staying unmatched (or being matched to an outside option). If $0 P_s c$ then school c is *unacceptable* to student s, and otherwise the school is *acceptable* to s. For c, $c' \in C$ we write cP_sc' if student s strictly prefers school c to school c', and cR_sc' if either cP_sc' or $c = c'$. Let $P_s \in (c)$ denote that student s has one acceptable school only, namely c, and let $P_s \in (0)$ denote that s has no acceptable schools. More generally, we will write $P_s \in (a, b, c)$, for example, to indicate that student s ranks school a first, b second, and c third, and that these are the only acceptable schools to s. We will also use the notation $r_s(c)$ for student s's ranking of school c for each acceptable school c. For example, $r_s(c) = k$ indicates that c is ranked in the kth position by s. Note that if $r_s(c) < r_{\hat{s}}(c)$ then s ranks school c *higher* than $\hat{s}$, in the sense that a lower rank number indicates higher preference. Let $\mathcal{P}_s$ denote the set of all preference relations for student $s \in S$ and let $\mathcal{P} = \mathcal{P}_{s_1} \times \ldots \times \mathcal{P}_{s_n}$. A preference profile is $P = (P_{s_1}, \ldots, P_{s_n})$, where $P \in \mathcal{P}$.

Each school $c \in C$ has a strict priority ordering $\succ_c$ of students in S. Let π be the set of all priority orderings (i.e., permutations) of students. Then for all $c \in C$, $\succ_c \in \pi$. Let $\Pi = \pi \times \ldots \times \pi$ be the m-fold Cartesian product of π. A priority profile is $\succ = (\succ_{c_1}, \ldots, \succ_{c_m})$, where $\succ \in \Pi$. We assume that a fixed strict priority profile $\succ \in \Pi$ is a primitive of the model. We will discuss in Section 9 how we can relax this assumption and give a different interpretation to Preference Rank Partitioned matching rules when coarse priorities are the primitives instead of strict priorities.

The outcome of a matching problem is an assignment of students to schools, which we refer to as a matching. Formally, a **matching** is a function $\mu : S \to C$, where $\mu(s) \in C$ indicates the school to which student s is matched. If a student s is unassigned in matching μ, we will write $\mu(s) = s$. For ease of notation we let μ_s denote $\mu(s)$ and μ_c denote $\mu^{-1}(c)$, the set of students assigned to c. For all $c \in C$, $|\mu_c| \leq q_c$, that is, the school capacity q_c cannot be exceeded. Let the set of matchings be denoted by M. A **matching rule** φ assigns a matching to each priority and preference profile pair $(\succ, P)$, or a *profile* for short. Thus, $\varphi : \mathcal{P} \times \Pi \to M$.

A matching μ is **blocked by student** $s \in S$ at $P \in \mathcal{P}$ if s prefers being single to being matched to μ_s, that is, $sP_s\mu_s$. A matching is **individually rational** at P if it is not blocked by any student at P. A matching μ is **non-wasteful** at P if no student s prefers a school to μ_s which has empty seats at μ, that is, for all $s \in S$ and $c \in C$, if $cP_s\mu_s$ then $|\mu_c| = q_c$.

Student s has **justified envy** at μ, given a profile $(\succ, P)$, if there exist school c and student $\hat{s}$ such that $cP_s\mu_s$, $s \succ_c \hat{s}$ and $\mu_{\hat{s}} = c$. That is, student s has justified envy for c, given that $\hat{s}$ is matched to c and $\hat{s}$ has lower priority for c than s. A matching μ is **stable** at $(\succ, P)$ if it is individually rational, non-wasteful, and there is no student who has justified envy at μ, given $(\succ, P)$. A matching rule is stable if it assigns a stable matching to each profile $(\succ, P)$.

A matching μ is Pareto-efficient if there is no $\eta \in M$ which Pareto-dominates μ, considering the student's preferences. A matching $\eta \in M$ Pareto-dominates μ if for all $s \in S$, $\eta_s R_s \mu_s$ and, for some $\hat{s} \in S$, $\eta_{\hat{s}} P_{\hat{s}} \mu_{\hat{s}}$. A matching rule is Pareto-efficient if it assigns a Pareto-efficient matching to each preference profile $(\succ, P)$. A matching is **optimal** if it is stable and Pareto-dominates all other stable matchings for the set of students. By Gale and Shapley (1962), there is a unique optimal stable matching for students at each profile. A matching rule is optimal if it assigns the optimal matching for students to each preference profile.

3 Preference Rank Partitioned Matching Rules

We now describe the family of matching rules that we study in this paper, called Preference Rank Partitioned (PRP) matching rules. Each PRP rule is determined by a profile of "partitions" of both the priority rankings of schools and the preference rankings of students. Each school's priority rankings are partitioned by specifying the number of consecutively ranked students in each member of the partition, starting from the top of the rankings.[2] Each student's preference rankings are also partitioned similarly by specifying the number of consecutively ranked schools in each member of the partition.

PRP rules are choice-based Deferred Acceptance rules, that is, each school uses a choice function to select among applicants in each round of the DA procedure. For each school $c \in C$, we define a choice function Ch_c such that for all $S' \subseteq S$, $Ch_c(S') \subseteq S'$ with the following properties. If $|S'| \leq q_c$ then $Ch_c = S'$ and if $|S'| \geq q_c$ then $|Ch_c(S')| = q_c$. We may also write $Ch_c(S', (\succ, P))$, if we want to indicate explicitly that Ch_c depends not just on S' but also on the profile $(\succ, P)$.

Choice-Based Deferred Acceptance rules:

- **Round 1:**

  Each student applies to her highest-ranked school (assuming that the highest-ranked school is acceptable to the student). Each school tentatively assigns its seats according to its choice function. Any remaining applicants are rejected.

- **Round *k*:**

  Each student who was rejected in round $k-1$ applies to her next highest-ranked acceptable school (if any remains). Each school considers the students who are tentatively assigned to the school, if any, together with its new applicants (henceforth the *applicant pool*), and tentatively assigns its seats according to its choice function. Any remaining applicants are rejected.

The algorithm terminates when each student is either tentatively assigned to some school or has been rejected by each school that is acceptable to the student, in which case the student remains unassigned.

A PRP rule is a choice-based DA rule with a choice function Ch_c for each school $c \in C$ which selects among students as a function of the given partitions of priorities and

[2] A partition of priority rankings may arise naturally when priorities are coarse, as is often the case in school choice, but here we treat the partitions as part of the matching rule, given the strict priorities for each school. We will discuss how to start from naturally arising coarse priorities in Section 9.

preferences. Given a partition of the priority rankings for school c, the **priority (rank) classes**, and given a partition of the preferences for each student, the **preference (rank) classes**, school c first selects students from its applicant pool in its highest priority class(es). If this does not determine the selected set of students, where the selected number of students is up to the fixed capacity q_c of the school, given that priority classes are not necessarily singletons, the choice function then considers the partitioned preferences, and selects students who have school c in their highest possible preference class(es) relative to each other. If the preference partitions still do not determine the selected set of students for school c, then the choice is resolved based on the strict priority ordering $\succ_c$. This last round based on the strict priority orderings can be seen as tie-breaking. This defines a choice function for each school $c \in S$, that is, it determines unambiguously the set of selected students up to the school capacity from any given applicant pool $S' \subseteq S$ for each $\succ_c$ and preference profile P. This, in turn, defines a PRP matching rule.

In sum, the school choice functions select students from the applicant pool lexicographically in the following order:

1. based on the **priority classes**;
2. based on the **preference classes**;
3. based on the **tie-breaker** given by the strict priority ordering.

Given their central role in the definition of PRP rules, we now define priority and preference rank classes formally. For all $c \in C$, let the cardinalities of the priority rank classes be denoted by $v_c^1, v_c^2, \ldots$, starting with the top-ranked students, such that $\sum_t v_c^t = n$. The **coarsest priority rank partition** is when $v_c^1 = n$ includes all students and the partition has one member only, and the **finest priority rank partition** is given by $(v_c^1, \ldots, v_c^n) = (1, \ldots, 1)$ with n classes, where each class contains one student. More generally, the rank partition is the finest or coarsest, respectively, whenever the resulting matching rule is outcome equivalent with the above. Given that a school capacity may be greater than one, this implies that a priority rank partition for school c is the finest if the students in priority ranks $q_c + 1, q_c + 2, \ldots, n$ are in their own singleton priority classes, and the priority classes of ranks 1 to q_c are irrelevant (may or may not consist of singletons).

Given a priority profile $\succ$, for all $c \in C$, let $\succ_c^1$ be the set of students ranked by $\succ_c$ between 1 and v_c^1 and, for all $t \geq 2$, let $\succ_c^t$ be the set of students ranked by $\succ_c$ between

$\sum_{l=1}^{t-1} v_c^l + 1$ and v_c^t. Note that for all $t \geq 1$, $| \succ_c^t | = v_c^t$. Let $v_c = (v_c^1, \ldots)$ denote the priority rank class list for school $c \in C$, and let $v = (v_c)_{c \in C}$ be the priority rank class profile.

For all $s \in S$, let the cardinalities of the preference rank classes be denoted by $p_s^1, p_s^2, \ldots$, starting with the top-ranked schools, such that $\sum_t p_s^t = m$. The **coarsest preference rank partition** is when $p_s^1 = m$ includes all schools and the partition has one member only, and the **finest preference rank partition** is given by $(p_s^1, \ldots, p_s^m) = (1, \ldots, 1)$ with m classes, where each class contains one school. Given our general remark above, technically the preference rank partition is also coarsest if having more than one preference class always leads to the same matching as having just one. For example, if $n \leq \sum_{c \in C} q_c$ then no student is rejected by her mth-ranked school, and thus $p_i^1 = m - 1$ also yields a coarsest partition.

Given a preference profile P, for all $s \in S$, let P_s^1 be the set of *acceptable* schools ranked by P_s between 1 and p_s^1 and, for all $t \geq 2$, let P_s^t be the set of *acceptable* schools ranked by P_s between $\sum_{l=1}^{t-1} p_s^l + 1$ and p_s^t. Note that for all $t \geq 1$, either $|P_s^t| = p_s^t$, if all schools are acceptable to student s, or if some schools are unacceptable then there exists $\hat{t}$ such that for all $l = 1, \ldots, \hat{t} - 1$, $|P_s^l| = p_s^l$ and $|P_s^{\hat{t}}| < p_s^{\hat{t}}$, and for all $l > \hat{t}$, $P_s^l = \emptyset$. Let $p_s = (p_s^1, \ldots)$ denote the preference rank class list for student $s \in S$, and let $p = (p_s)_{s \in S}$ be the preference rank class profile.

Using the above notation, each PRP matching rule is determined by a priority and a preference rank class profile (v, p). This is not to be confused with the profile $(\succ, P)$, a pair of a strict priority profile and a strict preference profile, which determines a specific problem and is a primitive of our model, while (v, p) is part of the matching rule. We will indicate explicitly the priority and preference rank class profiles for a PRP matching rule and denote it by $f^{v,p}$. Thus, $f^{v,p}(\succ, P) = \mu$ indicates that the PRP rule $f^{v,p}$ assigns matching μ to profile $(\succ, P)$.

We can now formally define the choice function Ch_c for each school $c \in C$ for a PRP rule $f^{v,p}$. Fix $c \in C$ and let $S' \subseteq S$. As already stated, the set of students $T \subseteq S'$ is selected from applicant pool S' based on priority rank classes first, then based on preference rank classes, and finally based on the tie-breaker given by strict priorities. Formally, if $|S'| > q_c$ then $Ch_c(S') = T$ if the following are satisfied:

- $T \subset S'$, $|T| = q_c$

- there exists $k \geq 1$ such that for all $s \in T$, $s \in \bigcup_{t=1}^{k} \succ_c^t$;
- for all $\hat{s} \in S' \setminus T$, if $\hat{s} \in \bigcup_{t=1}^{k} \succ_c^t$ then $\hat{s} \in \succ_c^k$;
- for all $s \in T$ and $\hat{s} \in S' \setminus T$ such that $s, \hat{s} \in \succ_c^k$, if there is no $k' \geq 1$ such that $c \in \bigcup_{t=1}^{k'} P_s^t$ and $c \notin \bigcup_{t=1}^{k'} P_{\hat{s}}^t$, then $s \succ_c \hat{s}$.

Example 1 (PRP choice functions). Consider the following matching problem with four students and four schools ($n = m = 4$). The preferences are given in the table below, in which the bars indicate the preference rank classes for the PRP rule. That is, $p_1 = (1, 2, 1), p_2 = (1, 3), p_3 = (3, 1)$, $p_4 = (1, 3)$.

P_1	P_2	P_3	P_4
b	b	d	b
c	a	b	a
a	d	a	d
d	0	c	c

Let $\succ_a = (1, 2, 3, 4)$ indicate the strict priorities in descending order for school a, and let $v_a = (3, 1)$. Assume that school a has capacity 1. Let the applicant pool for school a be $\{1, 3, 4\}$. Student 4 is eliminated based on the priority rank classes of school a, since $1, 3 \in \succ_a^1$ and $4 \in \succ_a^2$. This leaves students 1 and 3. Student 3 is selected based on the preference rank classes, since 1 ranks a in the second highest preference class and 3 ranks a in the highest preference class: $a \in P_1^2$ and $a \in P_3^1$.

Now consider the same problem with a slightly different PRP rule, where student 3's preference rank classes are different: $p_3 = (2, 1, 1)$. Given applicant pool $\{1, 3, 4\}$ for school a, student 4 is eliminated based on the priority rank classes of school a, as before, and given the new preference rank classes for student 3, now the selection cannot be made between 1 and 3 based on the preference rank classes, since both rank a in their second highest preference class: $a \in P_1^2$ and $a \in P_3^2$. Thus, we apply the tie-breaker strict priority order $\succ_a$ and student 1 is selected, since $1 \succ_a 3$. $\square$

PRP choice functions satisfy standard properties of choice functions that are associated with choice-based DA rules, such as Acceptance, Monotonicity, Substitutability and Consistency (Ehlers and Klaus, 2016), but these properties only hold when the preference profile is fixed. This is in contrast to typical choice functions used in conjunction with the DA, such as when quotas are specified for different types of agents, and the choice function is not a function of the preferences. This is the most salient feature of PRP rules, namely, that the choice functions of the schools depend on the students' preferences, and specifically on the students' preference rank classes, which is why we call these matching rules Preference Rank Partitioned rules. Thus, in contrast to choice-based DA rules which use choice functions that are independent of the students' preferences, this feature of PRP rules accounts for the loss of strategyproofness (see Theorem 3 in Section 8).

If both the priority rank partition for each school and the preference rank partition for each student are the coarsest, then the PRP rule relies only on the strict priorities as tie-breakers, and therefore this rule is the standard Deferred Acceptance rule, which simply selects the top priority students from each applicant pool. Equivalently, we can let the priority rank partition be arbitrary. As long as the preference rank partition is the coarsest, the PRP rule is the DA. Hence, as there may be multiple representations (v, p) of the same PRP rule, we will use the convention that the role of the tie-breaker should be minimized as much as possible by making the priority partition finer. As a consequence, given a finer priority partition, the preference partition should be left as coarse as possible, which clarifies the impact of the preferences. This specification of a PRP rule delineates which information is used by the choice function, whether it is the priority partition or the preference partition, and specifically which rank classes may play a role in any particular selection. In the case of the standard DA rule, this means that we let the priority partition be the finest for each school, so as to entirely eliminate tie-breaking, which then clarifies that the preference partitions don't play any role in the choice function, since all selections can be made based on the priority partitions, and therefore we can let each student's preference partition be the coarsest. Consequently, this convention not only lets us have a unique representation of each PRP rule, but it also provides an intuitive representation.

Formally, we can find the unique representation of a PRP rule f based on this convention as follows. Fix the priority profile $\succ \in \Pi$. Let $c \in C$ and $s, \hat{s} \in N$ such that $\hat{s} \succ_c s$. If there exists a preference profile $P \in \mathcal{P}$ such that $f_s(\succ, P) = c$ and $cP_{\hat{s}} f_{\hat{s}}(\succ, P)$ then let s and $\hat{s}$ be in the same priority class, and let all students $\tilde{s}$ such that $s \succ_c \tilde{s} \succ_c \hat{s}$ be also in the same priority class, and otherwise let each student for school c be in a separate priority class. This determines the priority rank partition profile v which satisfies the convention that v_c is as fine as possible for each school c. Note that since f is a PRP rule, we would get the same v using any priority profile $\succ \in \Pi$.

In order to determine the preference rank partition profile p, let $c \in C$ such that there is at least one priority class according to v_c with a minimal size of 2, that is, the priority class in v_c contains at least two ranks: $v_c^k \geq 2$ for some $k \geq 1$. If there is no such a school then all school priority partitions v_c are the finest and all preference partitions p_s are the coarsest, and f is the DA. Let students $s, \hat{s} \in S$ occupy these two ranks in the same priority class of $v_c : s, \hat{s} \in \succ_c^k$. Then, from the construction of v, there exists a priority profile $\succ \in \Pi$ with $\hat{s} \succ_c s$ and a preference profile $P \in \mathcal{P}$ such that $f_s(\succ, P) = c$ and $cP_{\hat{s}} f_{\hat{s}}(\succ, P)$. Find such a profile $(\succ, P)$ which maximizes $r_s(c)$ and minimizes $r_{\hat{s}}(c)$. Let $r_s(c) \in p_s^t$ and $r_{\hat{s}}(c) \in p_{\hat{s}}^{\hat{t}}$, where $t + 1 = \hat{t}$. If we repeat the same exercise for pairs of students $s, \hat{s}$ with an arbitrary school c for different profiles $(\succ, P)$ that meet the above specifications in terms of v_c then we can trace out the preference rank classes for each student and get a unique preference rank class profile p for the fixed PRP rule f such that p satisfies the convention that p_s is as coarse as possible for each student s, given the uniquely specified priority rank class profile v which is as fine as possible.

4 Special Subclasses of PRP Matching Rules

As already noted, the standard DA rule is a PRP rule, which is described by the finest priority partition profile and the coarsest preference partition profile. Another well-studied PRP rule besides the DA rule is the Boston (Immediate Acceptance) rule (Abdulkadiroğlu and Sönmez, 2003). The Boston rule is a PRP rule with the coarsest priority rank partition and selects among students based on the finest preference rank partition, hence tie-breaking is only necessary when students have the same ranking for a school: if students s and $\hat{s}$ are competing for school c then s is chosen over $\hat{s}$ if s ranks a better than $\hat{s}$ (i.e., $r_s(c) < r_{\hat{s}}(c)$) or if s and $\hat{s}$ rank c equally (i.e., $r_s(c) = r_{\hat{s}}(c)$) and $s \succ_c \hat{s}$.

Previously studied classes of matching rules that belong to the set of PRP rules include the Application-Rejection rules (Chen and Kesten, 2017) used in China, the

Table 1: Special members and subclasses of PRP matching rules

PRP Matching Rules	**Priority Partition**	**Preference Partition**
Deferred Acceptance (DA)	Finest	Coarsest
Boston	Coarsest	Finest
Deferred Boston	Homogeneous	Finest
First Preference First	Equal-preference schools: finest Preference-first schools: coarsest	Finest
Secure Boston	For each school c: finest for top q_c, then coarsest	Finest
French Priority	Arbitrary	Finest
Application-Rejection	Coarsest	Homogeneous
Taiwan Deduction	Coarsest	Homogeneous
Homogeneous PRP	Homogeneous	Homogeneous
Equitable PRP	Arbitrary	Homogeneous
Favored Students	Coarsest	Favored students: coarsest Non-favored students: finest

First Preference First rules (Pathak and Sönmez, 2013) that were banned in England, the Secure Boston rules and their generalizations proposed by Dur et al. (2019) to replace the Boston rule, and a special member of the French Priority rule introduced by Bonkoungou (2019), which corresponds to a broad class of PRP rules in our setting and includes all the previously mentioned rules (see more on this in Section 9), while in Bonkoungou's setup it is a single rule, based on the given coarse priorities.

We list some further notable subfamilies of PRP rules in Table 1 which have not been studied before. The Deferred Boston rules include both the standard DA and the Boston rules and allow for any homogeneous priority partition profile (i.e., the same priority partition for each school), while the preference partitions are the finest. If both the preference and priority partition profiles are homogeneous then we have a Homogeneous PRP rule. All Deferred Boston rules are Homogeneous PRP rules, but the French Priority rules in general are not Homogeneous PRP rules, and specifically the First Preference First and the Secure Boston rules are not Homogeneous PRP rules. On the other hand, the Application-Rejection rules are Homogeneous PRP rules. The class of Equitable PRP rules, characterized by a homogeneous preference partition profile, is even larger than the class of Homogeneous PRP rules, and contains all of the above mentioned rules and families of PRP rules. Lastly, to identify a specific class of PRP rules which does not belong to the class of Equitable PRP rules, we included in the table the family of Favored Students rules, which allow for different treatments of students. Specifically, Favored Students rules have the coarsest priority partition profile, and each student is either favored or not. Favored students have the coarsest preference partition, while not favored students have the finest. Favored Students and similar rules, which distinguish among students, may be desirable if one of the objectives of the matching is to prioritize certain classes of students, for example when affirmative action or equal opportunity policies are employed.

Although we defined PRP matching rules by using first the priority rank partition when selecting students, note that for the PRP matching rules that have the coarsest priority partition the priority rankings do not play any role in the selection of students up front (such as the Boston rule and more generally the Application-Rejection rules), and we understand intuitively that these rules make selections based on the preference rank partitions primarily, and the strict priorities are used for tie-breaking only when needed. In general, the PRP rules which don't have the coarsest priority partition profile, such as the Deferred Boston rules (excluding the Boston rule) or the First Preference First rules, the priority rank partitions play a role in student selection.

5 Stability and Optimality of PRP Matching Rules

The dependence of PRP choice functions on student preferences, which is the most notable general feature of PRP rules, accounts for violating typical stability conditions that are independent of the preferences. Given that when preference partitions are the coarsest the preferences play no role in choosing among applying students, and given that the only such PRP rule is the standard DA, this is the only rule which satisfies the standard stability axiom in the class of PRP matching rules.

We will now consider a stability concept inspired by PRP rules, which we call rank-partition stability. Given the rank partition profiles (v, p), for each profile $(\succ, P)$ we construct a strict priority profile $\bar{\succ}$ as follows. For each school c the orderings of students across priority rank classes based on $\succ_c$ and v_c remain the same in $\bar{\succ}_c((\succ, P), (v, p))$, and within priority rank classes we order students according to the preference rank partition of P based on p. If ties remain, then we use the strict priority ordering $\succ_c$ as a tie-breaker. More formally, let $s, \hat{s} \in S$, let $k, k' \geq 1$ such that $s \in \succ_c^k, \hat{s} \in \succ_c^{k'}$, and let $t, t' \geq 1$ such that $c \in P_s^t$ and $c \in P_{\hat{s}}^{t'}$. If $k \neq k'$ then s and $\hat{s}$ are in different priority rank classes and $s \tilde{\succ}_c \hat{s}$ if and only if $s \succ_c \hat{s}$. If $k = k'$ then s and $\hat{s}$ are in the same priority rank class. Then, if $t \neq t'$ then if $t < t'$ then $s \bar{\succ}_c \hat{s}$, and if $t > t'$ then $\hat{s} \bar{\succ}_c s$. Finally, if $k = k'$ and $t = t'$ then $s \bar{\succ}_c \hat{s}$ if and only if $s \succ_c \hat{s}$. Note that the priority profile $\bar{\succ}$ is a function of the preference profile P and thus it can change as preferences vary. From now on we will refer to $\bar{\succ}(\succ, P(v, p))$ as the *constructed priority profile.*

A matching rule φ is **rank-partition stable** if there exists a pair of rank partitions (v, p) such that $\varphi(\succ, P)$ is stable with respect to the constructed priority profile $\bar{\succ}((\succ, P), (v, p))$ at each profile $(\succ, P)$ and, for all $\succ, \succ' \in \Pi$, if $\bar{\succ}((\succ, P), (v, p)) = \bar{\succ}((\succ', P), (v, p))$ then $\varphi(\succ, P) = \varphi(\succ', P)$. We will also say that a matching rule φ is rank-partition stable with respect to (v, p). If a matching rule assigns a matching to each pair of priority and preference profiles $(\succ, P)$ which is stable with respect to the constructed priority profile $\bar{\succ}((\succ, P), (v, p))$, and if the selection of a stable matching only depends on this constructed priority profile at each preference profile then the matching rule is rank-partition stable.

We provide an example below to illustrate the construction of the preference-profile-specific constructed priority profile $\bar{\succ}(\succ, P(v,p))$.

Example 2 (**A constructed priority profile for rank-partition stability**)**.** Consider the following matching problem with five students and four schools ($n = 5, m = 4$). The preference profile P and the priority profile $\succ$ are given below, and the rank partitions (v,p) are specified by the bars in the two tables.

Student preferences

P_1	P_2	P_3	P_4	P_5
b	b	b	b	d
c	c	a	a	b
a	a	d	c	a
d	d	c	d	c

School priorities

$\succ_a$	$\succ_b$	$\succ_c$	$\succ_d$
4	3	4	4
1	1	1	5
2	4	5	3
3	2	2	1
5	5	3	2

Constructed priority profile $\bar{\succ}$

$\bar{\succ}_a$	$\bar{\succ}_b$	$\bar{\succ}_c$	$\bar{\succ}_d$
1	3	1	5
2	1	4	4
4	4	5	3
5	2	2	1
3	5	3	2

□

It is easy to see that each PRP rule is rank-partition stable. Moreover, it is straightforward to verify that each PRP rule is optimal within the set of rules that are rank-partition stable with respect to a given (v,p) due to a classic result by Gale and Shapley (1962), which implies in our setting that for each pair of priority and preference rank partition profiles (v,p) and for each profile $(\succ, P)$ there exists a rank-partition stable matching which is stable with respect to the priorities in the constructed priority profile $\bar{\succ}(\succ, P(v,p))$ for (v,p) at $(\succ, P)$, and each student weakly prefers this matching to any other matching which is stable with respect to $\bar{\succ}(\succ, P(v,p))$ at preference profile P. We call this matching the $(\mathbf{v},\mathbf{p})$**-optimal matching** at $(\succ, P)$. On can easily verify that the PRP rule $f^{v,p}$ is rank-partition stable with respect to (v,p) and $f^{v,p}(\succ, P)$ is the (v,p)-optimal matching at each profile $(\succ, P)$. We summarize these findings below.

Proposition 1. *Each PRP rule $f^{v,p}$ is rank-partition stable and selects the unique (v,p)-optimal rank-partition stable matching at each profile $(\succ, P)$.*

Rank-partition stability can be seen as a straightforward stability property of PRP rules, based on stability with respect to the appropriately constructed priority profile at each preference profile. A similar property is used by Bonkongou (2019) for French Priority rules. This representation of PRP rules and the underlying stability concept serve as a foundation for later results, as they highlight the parallel features between PRP rules and the standard DA rule, and allow us to see the PRP rules as optimal rules within the set of rank-partition stable rules, which are stable with respect to the modified priority profile at each preference profile, where the modifications of the priorities correspond to the selections made by PRP choice functions. Essentially, Proposition 1 tells us that a PRP rule $f^{v,p}$ can be seen as a DA rule with preference-profile-specific (v,p)-modified responsive priorities. Note also that the proposition implies that PRP rules are individually rational and non-wasteful.

6 Equitable PRP Matching Rules

Not all PRP matching rules treat students symmetrically with respect to their preferences. We call the subfamily of PRP rules which treat students symmetrically in terms of their preference partitions **Equitable PRP rules**, which can be described in terms of a homogeneous partition of preferences across students (i.e., the preference partition for each student is the same). We call these rules equitable since, for example, if student s has a coarser preference partition than student $\hat{s}$ then s gets a preferential treatment compared to student $\hat{s}$, given a fixed priority profile $\succ$. Although a PRP rule with the same preference partition for each student treats students equitably in terms of their preference rankings, we note that these rules only treat the preferences of students equally, but students may still be treated differently based on the school priority partition profile v.

Formally, a preference rank partition profile p is homogeneous if for all $s, s' \in S$, $p_s = p_{s'}$. Since the priority and preference rank partitions that describe a PRP rule are not always unique, as already discussed, we can follow the specified convention to determine if a PRP rule is an Equitable PRP rule. More generally, the requirement is that *there exists at least one* homogeneous preference rank partition (along with a priority rank partition) which yields the PRP rule. In other words, no matter which homogeneous preference rank partition profile and which arbitrary priority rank partition profile are used, for at least one pair of preference and priority rank partition profile (v, p) the outcome does not correspond to the outcome prescribed by the PRP rule.

To aid our analysis, we propose a general stability property of matching rules, called PP-stability, which weakens the standard stability axiom by basing the matching on a comparison of students' preference ranks of a school that they compete for, in addition to the students' priority rankings by this school. Characterizations of the Boston rule, such as the ones given by Kojima and Ünver (2014) and Doğan and Klaus (2018), rely on axioms comparing the preference ranking of alternatives, which are similar to the idea for this stability concept, since the Boston rule makes matches primarily based on the preference rankings. Our concept combines a priority-based and a preference-based justification for students to have justified envy, and this stronger justified envy is ruled out by our new stability concept. We can readily see that this is a stronger justified envy concept, since it is justified on the grounds that neither the priorities nor the preference rankings can explain the selection of one student over another at a school to which both students have applied. Afacan (2013) uses a similar property combining priority and preference rankings, but his axiom makes explicit use of the school capacity.

Preference and Priority Rank Stability (PP-Stability): Student s has **PP-justified envy** at μ, given $(\succ, P)$, if there exist school c and student $\hat{s}$ such that $cP_s\mu_s$, $s \succ_c \hat{s}$, $r_s(c) \leq r_{\hat{s}}(c)$, and $\mu_{\hat{s}} = c$. Student s has PP-justified envy for c, given that $\hat{s}$ is matched to c and $\hat{s}$ has both lower priority for c than s and ranks c the same or higher than $\hat{s}$. A matching μ is **PP-stable** at $(\succ, P)$ if it is individually rational, non-wasteful, and there is no student who has PP-justified envy at μ, given P. A matching rule is PP-stable if it assigns a PP-stable matching to each profile $(\succ, P)$.

It may appear at first that all PRP rules are PP-stable, but this is not the case. The following example shows a Favored Students PRP rule which is not PP-stable.

Example 3 (**A Favored Students PRP rule which is not PP-stable**)**.** Consider the following matching problem with five students and four schools $(n = 5, m = 4)$. Schools a, b, c have a capacity of one, and school d has a capacity of two. The preferences and priorities are below, with bars indicating the preference partitions in the preference profile.

Student preferences

P_1	P_2	P_3	P_4	P_5
b	b	b	b	d
c	c	a	a	b
a	a	d	c	a
d	d	c	d	c

School priorities

$\succ_a$	$\succ_b$	$\succ_c$	$\succ_d$
4	3	1	4
1	1	2	1
2	4	3	5
3	2	4	2
5	5	5	3

In this example students 1 and 2 are so-called favored students, which means that there is only one preference rank class for these students, which includes all the schools (the coarsest preference rank partition), while the non-favored students, 3, 4, and 5, have the finest preference rank partition: $p_1^1 = p_2^1 = 4$, and $p_s = (1, 1, 1, 1)$ for $s \in \{3, 4, 5\}$.

The rounds of the specified Favored Students rule at the given profile are summarized in the table below, with the selected students in each round underlined.

Round	a	b	c	d
1		$1, 2, \underline{3}, 4$		$\underline{5}$
2	$\underline{4}$	$\underline{3}$	$\underline{1}, 2$	$\underline{5}$
3	$\underline{2}, 4$	$\underline{3}$	$\underline{1}$	$\underline{5}$
4	$\underline{2}$	$\underline{3}$	$\underline{1}, 4$	$\underline{5}$
4	$\underline{2}$	$\underline{3}$	$\underline{1}$	$\underline{4}, \underline{5}$

In this example $r_4(a) = 2 < 3 = r_2(a)$ and $4 \succ_a 2$. Since aP_4d and $\mu_2 = a$, this rule is not PP-stable. $\square$

First we verify that PP-stability is independent of rank-partition stability. This is not surprising, but since both concepts weaken the standard stability axiom, it is in order. The Favored Students rules are rank-partition stable PRP rules by Proposition 1, but are not PP-stable, as seen in Example 3. A matching rule which assigns a non-PRP matching (e.g., the outcome of the Top Trading Cycles rule) at each profile $(\succ, P)$ where this matching is PP-stable and otherwise assigns the DA outcome is PP-stable but not rank-partition stable.

We remark that both the DA and Boston rules are PP-stable. Since stability implies PP-stability, the DA is clearly PP-stable. The Boston rule is also PP-stable, since if either $r_s(c) < r_{\hat{s}}(c)$ or if $r_s(c) = r_{\hat{s}}(c)$ and $s \succ_c \hat{s}$, in both cases $\hat{s}$ cannot be assigned to c unless s is also assigned to school $\tilde{c}$, whenever $\tilde{c}R_s c$. Indeed, the Boston rule is preference-rank stable in the sense that whenever $r_s(c) < r_{\hat{s}}(c)$, s does not envy $\hat{s}$ when $\hat{s}$ is assigned c. PRP rules in general are not preference-rank stable in this stronger sense. In fact, it is easy to check that the Boston rule is the only such preference-rank stable rule within the class of PRP rules. On the other hand, the DA rule is the only priority-rank stable (i.e., stable) rule in the class of PRP rules.

While not all PRP rules satisfy PP-stability, as demonstrated by Example 3, the subfamily of PRP rules which satisfy PP-stability is much larger than just the DA and Boston rules, and we will show in the next proposition that it exactly corresponds to the Equitable PRP rules. We remark that all previously studied PRP rules are Equitable PRP rules. Proposition 2 below provides an explanation for this, since it demonstrates that all the studied rules are PP-stable, which is an intuitive feature of matching rules and an attractive attribute in school choice.

Proposition 2. *A PRP rule is PP-stable if and only if it is an Equitable PRP rule.*

Proof.
Claim 1: An Equitable PRP matching rule is PP-stable.
Let $f^{v,p}$ be an Equitable PRP rule. That is, $f^{v,p}$ is a PRP rule and p is homogeneous across students. Suppose that there exists a profile $(\succ, P)$ such that $f^{v,p}(\succ, P)$ is not PP-stable. Then there exist $s, \hat{s} \in S$ and $c \in C$ such that $r_s(c) \leq r_{\hat{s}}(c)$ at P, $s \succ_c \hat{s}$, and s envies $\hat{s}$ at $(\succ, P)$ for being assigned to c. That is, $f^{v,p}_{\hat{s}}(\succ, P) = c$ and $cP_s f^{v,p}_s(\succ, P)$.

Let $t, t' \geq 1$ such that $c \in P^t_s$ and $c \in P^{t'}_{\hat{s}}$. Then, given $r_s(c) \leq r_{\hat{s}}(c)$, since p is homogeneous across agents, $t \leq t'$. Let $k, k' > 0$ such that $s \in \succ^k_c$ and $\hat{s} \in \succ^{k'}_c$. Then $s \succ_c \hat{s}$ implies that $k \leq k'$. Given that $f^{v,p}_{\hat{s}}(\succ, P) = c$ and $cP_s f^{v,p}_s(\succ, P)$, $k \leq k'$ implies that $k = k'$. Then $t \leq t'$ implies that $t = t'$. Hence, the tie-breaker $\succ_c$ would be used by $f^{v,p}$ to select between s and $\hat{s}$ at school c, but $s \succ_c \hat{s}$ implies that $f^{v,p}_{\hat{s}}(\succ, P) \neq c$ when $cP_s f^{v,p}_s(\succ, P)$, which is a contradiction.

Claim 2: A PP-stable PRP rule is an Equitable PRP rule.
Let $f^{v,p}$ be a PP-stable PRP rule. Suppose that $f^{v,p}$ is not an Equitable PRP rule, that is, p is not homogeneous across students. Then there exist $(\succ, P) \in \mathcal{P}$, $c \in C$, $s, \hat{s} \in S$ and $t, t' \geq 1$ such that

1. $f^{v,p}_{\hat{s}} = c$ and $cP_s f^{v,p}_s(\succ, P)$

2. $r_s(c) \leq r_{\hat{s}}(c)$ at P

3. $t > \hat{t}$, where $c \in P_s^t$ and $c \in P_{\hat{s}}^{\hat{t}}$

4. s and $\hat{s}$ are in the same priority rank class for c, given $\succ_c$ and v_c : there exists $k \geq 1$ such that $s, \hat{s} \in \succ_c^k$.

Since $f^{v,p}$ is PP-stable , $\hat{s} \succ_c s$. Note that $f^{v,p}$ selects $\hat{s}$ over s for school c based on the preference rank classes, since $t > \hat{t}$, not based on priority rank classes, given that s and $\hat{s}$ are in the same priority rank class for c. The fact that $\hat{s} \succ_c s$ is irrelevant, since the choice is not based in the tie-breaking. Let $\bar{\succ}_c$ be the same as $\succ_c$ except for the position of student $\hat{s}$: place student s directly above $\hat{s}$ in $\bar{\succ}_c$, and leave all other orderings in $\bar{\succ}_c$ the same as in $\succ_c$.

Let $\bar{\succ}' \equiv (\bar{\succ}_c, \succ_{-c})$. Then $f^{v,p}(\succ', P) = f^{v,p}(\succ, P)$, as we will show. Note first that only $\succ_c$ is changed to $\bar{\succ}_c$, so the only difference can be in the selections made by school c. Note that since s and $\hat{s}$ were in the same priority class at $\succ_c$ and since v_c remains the same, the students in the priority rank classes of c have not changed. Specifically, s and $\hat{s}$ are still in the same priority class at $\bar{\succ}_c$, and so are all $\bar{s} \in S$ such that $\hat{s} \succ_c \bar{s} \succ_s s$. Thus, $\succ_c^k = \bar{\succ}_c^k$, and for all k', $\succ_c^{k'} = \bar{\succ}_c^{k'}$. Therefore, all selections of school c based on the priority rank classes are the same, and subsequently all selections based on preference rank classes are the same at $(\succ', P)$ and at $(\succ, P)$. Thus, Ch_c makes the same selections at $\bar{\succ}_c$ as at $\succ_c$, and $f^{v,p}(\bar{\succ}, P) = f^{v,p}(\succ, P)$, as desired. Therefore, $f^{v,p}(\bar{\succ}, P) = c$ and $cP_s f^{v,p}(\succ', P)$. Since $r_s(c) \leq r_{\hat{s}}(c)$ at P and $s \bar{\succ}_c \hat{s}$, $f^{v,p}$ is not PP-stable. This is a contradiction, which implies that $f^{v,p}$ is an Equitable PRP rule. □

We will say that a matching rule is **equitable-rank-partition stable** if the definition of rank-partition stability for this rule is satisfied based on (v, p), where p is homogeneous. Equivalently, if there is a homogeneous preference rank partition p across students, that can be used in the construction of the priority table $\tilde{\succ}((\succ, P), (v, p))$ at each profile $(\succ, P)$. Then the matching rule equitable-rank-partition stable. Clearly, equitable-rank-partition stability implies rank-partition stability of a matching rule.

The next result clarifies the relationship between rank-partition stability and equitable-rank-partition stability. The proof of this theorem and all subsequent omitted proofs are in the appendix.

Theorem 1. *A rank-partition stable matching rule is PP-stable if and only if it is equitable-rank-partition stable.*

It is not difficult to see that each Equitable PRP rule is equitable-rank-partition stable. As in the case of PRP rules, we can easily verify that the Equitable PRP rule $f^{v,p}$ is equitable-rank-partition stable, that is, rank-partition stable with respect to (v, p) where p is a homogeneous preference rank partition profile, and $f^{v,p}(\succ, P)$ is the (v, p)-optimal matching at each profile $(\succ, P)$. Thus, we can state a similar result to Proposition 1 for Equitable PRP rules: each Equitable PRP rule $f^{v,p}$ is equitable-rank-partition stable and selects the (v, p)-optimal rank-partition stable matching at each profile $(\succ, P)$. Therefore, Theorem 1 is a generalization of Proposition 2. We provided a separate proof of Proposition 2 to give some intuition for the result which cannot easily be gleaned from the proof of Theorem 1.

7 Efficiency of PRP Matching Rules

PRP rules select the optimal rank-partition stable matching at each profile, as seen in Section 5. This implies that if the selected matching is not Pareto-efficient, it can only be Pareto-dominated by a matching that is not rank-partition stable. We know that the DA matching is in general not Pareto-efficient (Gale and Shapley, 1962; Balinski and Sönmez, 1999) so PRP rules, which choose the DA matching for the constructed priority profile, as stated by Proposition 1, are generally not Pareto-efficient. One notable exception is the Boston rule, which is Pareto-efficient (Abdulkadiroğlu and Sönmez, 2003), due to the fact that it assigns the school seats based primarily on the student preferences, and uses the school priorities only for tie-breaking, and thus cannot be Pareto-dominated by a matching which is not rank-partition stable. One may therefore conjecture that the only Pareto-efficient PRP rule is the Boston rule, but it turns out that a somewhat larger set of PRP rules is Pareto-efficient. Namely, there may be one student whose preference rank partition is not necessarily the finest, and may be chosen arbitrarily, but all other students' preference partitions have to be the finest. We call this class of rules, including the Boston rule, Near-Boston rules.

Near-Boston rules are PRP rules such that:

1. each school has the coarsest priority partition;

2. there exists $s_j \in S$ such that each student $s \in S \setminus \{s_j\}$ has the finest preference partition (and student s_j has an arbitrary preference partition).

Theorem 2. *A rank-partition stable rule is Pareto-efficient if and only if it is a Near-Boston rule.*

Corollary to Theorem 2. *An equitable-rank-partition stable rule is Pareto-efficient if and only if it is the Boston rule.*

This is an immediate corollary to Theorem 2, since if the rule is equitable-rank-partition stable then the rule uses a homogeneous preference rank partition profile p, and thus Theorem 2 implies that all students have the finest preference rank partition. This corollary generalizes the result of Chen and Kesten (2017) which shows that only the Boston rule is Pareto-efficient within the class of Application-Rejection rules.

8 Incentive Properties of PRP Matching Rules

Since PRP rules are generally not strategyproof, it is important to study their incentive properties.

For matching rule φ, given a profile $(\succ, P)$, if there is a student $s \in S$ and an alternative preference ranking $P'_s \in \mathcal{P}_s$ such that $\varphi_s(\succ, (P'_s, P_{-s}))P_s\ \varphi_s(\succ, P)$ then s can **manipulate** φ at P via P'_s, and rule φ is **manipulable** at P. We will also say that s can manipulate at P to obtain school $\varphi_s(\succ, (P'_s, P_{-s}))$. If a rule is not manipulable at any preference profile then the rule is **strategyproof.**

The standard Deferred Acceptance rule is well-known to be strategyproof when only the students' incentives are taken into account (Dubins and Freedman, 1981; Roth, 1982). However, we can find examples of preference profiles where a PRP rule (other than the DA) is manipulable, and we prove a negative result for all PRP rules excluding the DA rule. This makes sense intuitively: the DA is the only strategyproof rule in the class of PRP matching rules, since it is the only PRP rule for which the school choice functions are independent of the preferences. This result extends a similar result by Chen and Kesten (2017) for Application-Rejection rules.

Theorem 3. *A rank-partition stable rule is strategyproof if and only if it is the Deferred Acceptance rule.*

Although PRP rules are not strategyproof in general, we can compare them based on their manipulability, using a simple comparison criterion that was first put forward by Pathak and Sönmez (2013) and subsequently studied by Chen et al. (2016).

Given a matching rule φ, for all students $s \in S$ and all profiles $(\succ, P)$, let $I(s, \varphi, (\succ, P)) = \{\varphi_s(\succ, (P'_s, P_{-s})) : P'_s \in \mathcal{P}_s,\ \varphi_s(\succ, (P'_s, P_{-s}))P_s\varphi_s(\succ, P)\}$. A matching rule φ is **less manipulable** than matching rule ψ if for all $s \in S$ and all profiles $(\succ, P)$, $I(s, \varphi, (\succ, P)) \subseteq I(s, \psi, (\succ, P))$ and there exists a profile $(\bar{\succ}, \bar{P})$ for which $I(s, \varphi, (\bar{\succ}, \bar{P})) \subset I(s, \psi, (\bar{\succ}, \bar{P}))$.

A PRP rule and all other rules which satisfy rank-partition stability with respect to the same (v, p) are comparable in terms of how vulnerable to manipulability they are according to the above definition, and the PRP rule stands out as the least manipulable in this class of rules. Formally, each PRP rule $f^{v,p}$ is less manipulable than any other matching rule which is rank-partition stable with respect to (v, p). A similar result has been obtained by Bonkoungou (2019) for the French Priority rules. These results follow from a general relationship between weak Pareto-domination and relative manipulability, as shown by Chen et al. (2016). According to their results, if a rule φ weakly Pareto-dominates another one rule ψ, then φ is less manipulable than ψ. Thus, the result follows from the optimality of PRP rules.

The following theorem is our main theorem on the incentive properties of PRP rules. The theorem says that when a PRP matching rule is used, a student cannot manipulate to obtain a seat at school c by placing c in the same or a lower preference rank class than the preference rank class where c belongs truthfully. This theorem has some interesting and wide-ranging implications for PRP rules, as we will explain below.

Theorem 4. *Let $f^{v,p}$ be a PRP rule and fix a profile $(\succ, P)$. Let $s \in S$ and $c \in C$ such that $cP_s f^{v,p}_s(\succ, P)$. Let $P'_s \in \mathcal{P}_s$ such that c is in the same or lower preference rank class in P'_s than in P_s, given p. Then $f^{v,p}_s(\succ, (P'_s, P_{-s})) \neq c$.*

By this theorem, a seat at a school can only be obtained by manipulation when reporting the school to be in a higher preference class than it truthfully is, regardless of what the reported preferences are otherwise. This gives a good idea about how the PRP rules are manipulable in general. The theorem also offers an intuitive explanation for two well-known results: why the DA rule is not manipulable, and why the Boston rule is so markedly manipulable (see, for example, Troyan and Morrill (2019)). Notably, given that the standard DA rule is the PRP rule with the coarsest preference partition for each student, this theorem implies that the Deferred Acceptance rule is strategyproof, since there is only one preference class for each student, and thus no school can be obtained by manipulation at any profile. At the other extreme, the Boston rule is the PRP rule with the finest preference partition for each student combined with the coarsest priority partition for each school (so priorities are only used to break ties), and thus the theorem sheds light on why the Boston rule is so manipulable: each change in the reported preferences results in placing at least one school in a higher preference class, and the top-ranked school is the only one which can never be obtained by manipulation when the Boston rule is used.

PRP rules between the DA and Boston rules are moderately manipulable, and the extent of manipulability depends on how coarse their preference partitions are. For example, the Application-Rejection rules have homogeneous preference rank classes, and larger preference rank classes would imply, based on the theorem, that there is generally less room for manipulation. Indeed, this result is shown by Chen and Kesten (2017), although this is not a direct implication of Theorem 4. When the preference rank partition is not homogeneous, that is, the PRP rule is not an Equitable PRP rule, the extent to which the PPR rule is vulnerable to manipulation varies with the student. In the extreme case of Favored Students rules, where each student has either the coarsest or the finest preference rank partition, it follows from the theorem that the favored students with the coarsest preference partitions cannot manipulate at all. However, the non-favored students, having the finest preference partitions, have lots of room to manipulate.

The following two results are implied directly by Theorem 4.

Corollary to Theorem 4. *Let $f^{v,p}$ be a PRP rule.*

1. *For all $s \in S$, if some school c is ranked in s's top preference rank class at some preference profile P, given p, then s cannot manipulate to obtain school c at P when using rule $f^{v,p}$.*

2. *For all students $s \in S$, if $p_s = (n)$ then student s cannot manipulate $f^{v,p}$ at any profile $(\succ, P)$.*

The first result in the corollary says that a school which is ranked in the top preference class of a student cannot be obtained as a result of successful manipulation by this student. This implies that a student cannot obtain his first-ranked school by manipulating any French Priority rule, and specifically the Boston rule. Our corollary is more general, since it allows for coarse preference partitions and thus for multiple schools in the top preference class that cannot be profitably obtained by misreporting the preferences. This, in turn, is further generalized by Theorem 4, since it also shows that trying to manipulate by ranking a school in a preference class that is either the same or below the preference class of the school to which a student would be matched otherwise is futile. Note that Theorem 4 is unrelated to the incentive analysis of Bonkoungou (2019), since he studies the given coarse priorities of French Priority rules, and compares the matching rules using a new criterion based on how fine the priority partitions are, while our results concern the preference partitions.

The second result in the corollary states that if a student has the coarsest preference partition then this student cannot manipulate any PRP rule. This is because in this case the student has every school in her top preference class. This is the case for all students in the DA rule, and this also holds for all favored students in Favored Students rules.

9 Coarse Priorities: An Extension

PRP matching rules can be extended naturally to the case where the given priorities are coarse (i.e., given by a weak order), as would be typical for New York City high schools (Abdulkadiroğlu et al., 2009). Another well-known example is Boston, where school priorities are set up based on older siblings attending the school and walk-zone priorities (Abdulkadiroğlu et al., 2005). In international refugee assignment, countries may have mandated priorities based on the level of danger faced by refugees, or may wish to prioritize specific skills or traits, which leads to coarse priorities, given the number of refugees who are looking for asylum (Jones and Teytelboym, 2017). If priorities are coarse, a strict priority ordering $\succ_c$ that is consistent with the given coarse priorities would be determined by a lottery for each school c and used for tie-breaking only if the choice function cannot make the required selection based on the coarse priorities and the preference rank partitions. Then the given coarse priorities for schools would be primitives of the model, not part of the specification of the rule, and the randomly selected strict priority profile $\succ$, which is consistent with the coarse priorities, would become part of the matching rule as a tie-breaker. This would imply that each member of the family of PRP rules is associated with the strict tie-breaker priority profile $\succ$ and the preference rank partition profile p. This is the setup of Bonkongou (2019), since universities in France only have weak priorities over students.

This lends additional applicability to PRP rules and makes the current study relevant to situations where coarse priorities arise naturally. We could further enhance the applicability of PRP rules by combining the primary and secondary interpretations (strict versus coarse priorities as primitives) in a natural manner, which would allow for coarse priorities as primitives but let the matching rule further partition the priority rankings within the priority rank classes given by the fixed coarse priorities, while breaking the ties randomly over the remaining weak priorities to create a tie-breaker. Our results extend to all these "hybrid" cases as well in a natural manner, but in the interest of a more accessible exposition we omit the formal presentation of this more general setup.

10 Conclusion

We have explored PRP rules, which are generalized Deferred Acceptance rules that allow students' preference rankings to play a role in the schools' choice functions, that is, the selection among competing applicants for a school is based partially on how high students rank the particular school in their preferences, and not only on their priority rankings by this school. Since we consider a large class of matching rules that includes many previously known rules, in addition to interesting new rules that are studied here for the first time, this paper offers a unified approach to these rules and establishes results that apply to all of them. As a foundation for our analysis that ties together all PRP rules, we show that all these matching rules can be understood as the DA matching selected on the basis of an appropriately modified priority profile at each preference profile, where the modifications of the priorities reflect how PRP school choice functions select students based on preference ranks (Proposition 1). Thus, each PRP rule is optimal in this sense, just like the Deferred Acceptance rule.

We have proved that the only Pareto efficient PRP rules are the so-called Near-Boston rules (Theorem 2), a class of rules which includes the Boston rule and other similar rules which only differ in the preference rank partition of one student from the

Boston rule. We have also shown that the only strategyproof PRP rule is the standard Deferred Acceptance rule (Theorem 3). These two results underline the difficulty of obtaining matching rules with both good incentives and strong efficiency properties, when it is desirable to allow the preferences of students to directly affect their chances of being accepted by the school, but the students' priorities are also taken into account.

We explore the classic tension among stability, efficiency, and incentive properties in our setup, which has been studied by a vast array of papers in various matching models and settings. Our main contribution to this literature is to show the specifics of this tension within the class of PRP rules, that is, when student selection by schools takes into account the preference rankings, and Theorems 2 and 3 can be best understood in terms of trade-offs. Namely, when the school priorities have a small impact (mainly used for tie-breaking), we get more efficiency, since the matching rule relies primarily on the preference rankings, as in the Boston rule and similar PRP rules. When the school priorities have a large impact, preference rankings play only a small role in the schools' selection, and the incentive properties are improved, but it is only in the extreme case of the DA rule, when the preferences have no impact on the school choice functions, that full strategyproofness can be achieved.

The main trade-off is between efficiency and incentives when rank-partition stability is required, a stability condition which allows matchings to be based on student preference rankings in addition to school priorities in a systematic manner. As we have seen, we get some extreme rules only in the family of PRP rules or, more broadly, among rank-partition stable rules, that satisfy efficiency (the Near-Boston rules) and, similarly, only the DA rule satisfies strategyproofness among PRP or rank-partition stable rules. The Boston and DA rules can be seen as the two extreme members of the family of PRP rules. The intuition is that as we place more emphasis on the preferences, we attain more efficiency and get closer to the Boston rule, and as we place more emphasis on the priorities, we get better incentives and get closer to the DA rule. This already gives a general idea to the designer about how to select among PRP rules, depending on the relative importance of the objectives: is efficiency more desirable, or are correct incentives more preferable? If efficiency comes first, choose coarser priority partitions and finer preference partitions, and if incentives are more important then choose finer priority partitions and coarser preference partitions.

One may argue, however, that incentives should always come first, given that if preferences are not reported accurately then efficiency cannot be enforced, since any normative criterion can only be based on the reported preferences, the only input into the mechanism regarding student welfare. But, especially in light of the substantial trade-offs, this argument also has some limitations, and thus the extent and specifics of manipulation are of considerable interest to the designer. Our main result on incentives (Theorem 4) provides guidance on this, as it sheds light on how students can manipulate PRP rules: we show that this is only possible by placing a desired but unattainable school in a higher preference rank class in the reported preferences than it is truthfully. This theorem also clarifies the different potential extent of manipulation for different PRP rules, both in terms of how and whether a school can be obtained by manipulation, depending on its preference rank classes as reported by different students, and also regarding the scope of manipulation which may differ widely among students, even for Equitable PRP rules.

Theorem 4 therefore provides insight and contributes to a more sophisticated understanding of PRP rules than afforded by the rough comparison of the Boston and DA rules themselves. Between the extremes of the DA and Boston rules, several different dimensions may be considered when searching for an appropriate matching rule, as evidenced by the fact that the family of PRP rules encompasses multiple subfamilies of PRP rules that bridge these same two matching rules, such as the First Preference First rules (or more generally the generalized Secure Boston rules), the Deferred Boston rules, the Application-Rejection (or Taiwan Deduction) rules and the Favored Students rules. Given that quite a few members of the class of PRP rules are used in real-life school choice and student placement systems, our theoretical analysis provides practically relevant policy insight. Therefore, our findings should be helpful for the design of assignment mechanisms that rely on the preference rankings directly when choosing among competing applicants.

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Appendix: Proofs

Proof of Theorem 1

Theorem 1. *A rank-partition stable matching rule is PP-stable if and only if it is equitable-rank-partition stable.*

Proof.
Claim 1.1: An equitable-rank-partition stable matching rule is PP-stable.

Proof: Let φ be an equitable-rank-partition stable matching rule. Suppose that there exists a profile $(\succ, P)$ such that $\varphi(\succ, P)$ is not PP-stable. Then there exist $s, \hat{s} \in S$ and $c \in C$ such that $r_s(c) \leq r_{\hat{s}}(c)$ at P, $s \succ_c \hat{s}$, and s envies $\hat{s}$ at $(\succ, P)$ for being assigned to c. That is, $\varphi_{\hat{s}}(\succ, P) = c$ and $cP_s\varphi_s(\succ, P)$. Let φ be equitable-rank-partition stable with respect to (v, p). Then p is homogeneous across students. Given p, let $t, t' \geq 1$

such that $c \in P_s^t$ and $c \in P_{\hat{s}}^{t'}$. Given v, let $k, k' \geq 1$ such that $s \in \succ_c^k$ and $\hat{s} \in \succ_c^{k'}$. Let $\tilde{\succ}$ denote the constructed strict priority profile $\tilde{\succ}((\succ, P), (v, p))$. Suppose $k < k'$. Then $s \tilde{\succ}_c \hat{s}$ and s has justified envy at $\varphi(\succ, P)$ based on $\tilde{\succ}_c$. This contradicts the fact that φ is equitable-rank-partition stable. Thus, $k \geq k'$. If $k > k'$ then $\hat{s} \succ_c s$ would follow, which is a contradiction. Therefore $k = k'$.

Now suppose $t < t'$. Then, since $k = k'$, $s \tilde{\succ}_c \hat{s}$, and s has justified envy at $\varphi(\succ, P)$ based on $\tilde{\succ}_c$. This again contradicts the fact that φ is equitable-rank-partition stable, and thus $t \geq t'$. Then, since $r_{s(c)} \leq r_{\hat{s}}(c)$ and p is homogeneous across students, it must be the case that $t = t'$. But then $s \succ_c \hat{s}$ implies that $s \tilde{\succ}_c \hat{s}$, since the tie-breaker is applied in the construction of $\tilde{\succ}_c$ when determining the relative positions of s and $\hat{s}$ in $\tilde{\succ}_c$. This implies that s has justified envy at $\varphi(\succ, P)$, based on $\tilde{\succ}_c$, which contradicts the fact that φ is equitable-rank-partition stable. Therefore, $\varphi(\succ, P)$ is PP-stable, and since this holds for all $(\succ, P)$, φ is PP-stable.

Claim 1.2: If a rank-partition stable matching rule is PP-stable then it is equitable-rank-partition stable.

Proof: Let φ be a rank-partition stable matching rule which is PP-stable. Let φ be rank-partition stable with respect to (v, p). Suppose that p is not homogeneous across students. Then there exist $(\succ, P)$, $c \in C$ and $s, \hat{s} \in S$ such that

1. $\varphi_{\hat{s}}(\succ, P) = c$ and $c P_s \varphi_s(\succ, P)$
2. $r_s(c) \leq r_{\hat{s}}(c)$ at P
3. $t > t'$, where $c \in P_s^t$ and $c \in P_{\hat{s}}^{t'}$
4. s and $\hat{s}$ are in the same priority rank class for c, given $\succ_c$ and v_c: there exists $k \geq 1$ such that $s, \hat{s} \in \succ_c^k$.

Let $\tilde{\succ}$ denote the constructed priority profile $\tilde{\succ}((\succ, P), (v, p))$. Since φ is PP-stable, given $r_{s(c)} \leq r_{\hat{s}}(c)$, we have $\hat{s} \succ_c s$. Since s and $\hat{s}$ are in the same priority rank class for c, given $\succ_c$ and v_c, and since $t > t'$, $\hat{s} \tilde{\succ}_c s$. Let $\succ'_c$ be the same as $\succ_c$, except for the positions of $\hat{s}, s$ and students ranked between $\hat{s}$ and s in $\succ_c$. Let $s \succ'_c \hat{s}$, and for all $\bar{s}$ such that $\hat{s} \succ_c \bar{s} \succ_c s$, if $c \in P_{\bar{s}}^{t'}$, let $\hat{s} \succ'_c \bar{s}$, and if $c \in P_{\bar{s}}^t$, let $\bar{s} \succ'_c s$. Leave all other orderings in $\succ'_c$ as in $\succ_c$ and specifically keep the set of students in the priority class of s and $\hat{s}$ the same.

Let $\succ' \equiv (\succ'_c, \succ_{-c})$. Let $\tilde{\succ}'$ denote the constructed priority profile $\tilde{\succ}'((\succ', P), (v, p))$. We will show that $\tilde{\succ}' = \tilde{\succ}$. Given that both are a function of $(\succ, P)$ and (v, p), and that only $\succ_c$ has changed, the only possible difference is between $\tilde{\succ}'_c$ and $\tilde{\succ}_c$, while $\tilde{\succ}'_{-c} = \tilde{\succ}_{-c}$.

We will show that $\tilde{\succ}'_{-c} = \tilde{\succ}_{-c}$. Given the difference between $\tilde{\succ}'_c$ and $\tilde{\succ}_c$, all students are still in the same priority rank class for school c at $\tilde{\succ}'_c$ as at $\tilde{\succ}_c$, and since the preference profile is unchanged, given $t > t'$ and $\hat{s}\tilde{\succ}'_c s$. Note that whenever the selection is based on the preference rank classes among the students in $\succ'^k_c$, the strict priority order $\succ'_c$ of the students within this set is irrelevant. Moreover, if the choice is based on the tie-breaker, the relevant orderings are preserved by construction. Thus, $\tilde{\succ}'_c = \tilde{\succ}_c$, which implies that $\tilde{\succ}' = \tilde{\succ}$. Since $s \succ'_c \hat{s}$ and $r_s(c) \leq r_{\hat{s}}(c)$ at $(\succ, P)$, PP-stability of φ implies that $\varphi(\succ', P) \neq \varphi(\succ, P)$. However, since $\tilde{\succ}' = \tilde{\succ}$, the rank-partition stability of φ implies that $\varphi(\succ', P) = \varphi(\succ, P)$, and we have a contradiction. Thus, p is homogeneous and φ is equitable-rank-partition stable. $\square$

Proof of Theorem 2

Theorem 2. *A rank-partition stable rule is Pareto-efficient if and only if it is a Near-Boston rule.*

Proof.

Claim 2.1: A Near-Boston rule is rank-partition stable and Pareto-efficient.

Proof: A Near-Boston rule is a PRP rule, and thus it is rank-partition stable by Proposition 1. We need to show that Near-Boston rules are Pareto-efficient. Let $f^{v,p}$ be a Near-Boston rule and fix a profile $(\succ, P)$. We will prove that $f^{v,p}(\succ, P)$ is Pareto-efficient.

Let $\mu \equiv f^{v,p}(\succ, P)$. Note that since $f^{v,p}$ is a Near-Boston rule, all priority partitions are the coarsest and there exists $j \in S$ such that for for all $s \in S \setminus \{j\}$, s's preference partition is the finest. Choose a permutation σ of the set of students S which follows the order of when the assignments are made in the $f^{v,p}$ procedure at $(\succ, P)$. Formally, for all $s \in S$, let t_s denote the round in the $f^{v,p}$ procedure at $(\succ, P)$ in which s is accepted by μ_s for the first time, and let σ be such that for all $i, l \in S$, $\sigma(i) \leq \sigma(l)$ if and only if $t_i \leq t_l$. It is well-known (see, for example, Balinski and Sönmez (1999) and Svensson (1999)) that if σ is a permutation of S such that if students choose their favorite school with remaining seats following the order specified by σ and the resulting matching is μ, that is, if σ is used as in a Serial Dictatorship f^{σ}, then μ is Pareto-efficient. Clearly, σ has this property if $f^{v,p}$ is the Boston rule, that is, if j has the finest preference partition along with all other students, since in this case there are no temporary assignments in the procedure: once a school accepts a student, the assignment is final. If σ does not have this property then there exist $i, l \in S$ such that $\sigma(i) < \sigma(l)$ and $\mu_l P_i \mu_i$. This implies that student i applied for school μ_l in some round $t < t_i$ and got rejected, which in turn implies that μ_l was already filled to capacity in round t. Since $t < t_i \leq t_l$ and l is assigned to μ_l in round t_l, it must be the case that $l = j$. Let $h \in S$ be the student who is assigned μ_j in round t, but $\mu_h \neq \mu_j$. Then h is selected over i in round t by school μ_j, but since in round t_j student j is assigned to μ_j, h is only temporarily assigned to μ_j and is rejected by μ_j in round t_j, where j is selected by μ_j over h. Note that h is unique since only p_j is a preference partition which is not the finest, and $h \neq i, j$.

Suppose, by contradiction, that μ is not Pareto-efficient. Then, given that whenever $i, l \in S$ are such that $\sigma(i) < \sigma(l)$ and $\mu_l P_i \mu_i$, we have $l = j$, as we have shown, there must be an envy chain from j to such a student i, that is, there exists student $\hat{i}$ who is assigned $\mu_{\hat{i}}$ in round $t_{\hat{i}}$, where $t_i \leq t_{\hat{i}} \leq t_j$ and $\mu_{\hat{i}} P_j \mu_j$. Note that $\hat{i} \neq j, h$ but $\hat{i} = i$ is possible. Thus, $r_j(\mu_{\hat{i}}) < r_j(\mu_j)$. Moreover, $\mu_{\hat{i}} \in P_{\hat{i}}^{t_{\hat{i}}}$, since $\hat{i} \neq h$, given that $t < t_{\hat{i}}$.

Since $p_i = p_h = (1, \ldots, 1)$, for all $c \in C$, $c \in P_i^{r_i(c)}$ and $c \in P_h^{r_h(c)}$. Moreover, in each round k where a student gets rejected by a school prior to round t_j, the student ranks the school in the kth position. Since h is selected over i in round t by μ_j, $r_i(\mu_j) = t$, $\mu_j \in P_i^t$ and thus, given that all school priorities are the coarsest, $\mu_j \in P_h^{t'}$, where $t' \leq t$.

Since $\mu_j \in P_h^{t'}$ and j is selected over h in round t_j by μ_j, we have $\mu_j \in P_j^{t''}$ such that $t'' \leq t'$, given that all the priority partitions are the coarsest. Then, since $r_j(\mu_{\hat{i}}) < r_j(\mu_j)$, $\mu_{\hat{i}} \in P_j^{\bar{t}}$ such that $\bar{t} \leq t''$. Observe that $\bar{t} \leq t'' \leq t' \leq t < t_i \leq t_{\hat{i}}$. Therefore, $\bar{t} < t_{\hat{i}}$. Since $\mu_{\hat{i}} \in P_{\hat{i}}^{t_{\hat{i}}}$ and $\mu_{\hat{i}} \in P_j^{\bar{t}}$, where $\bar{t} < t_{\hat{i}}$, it is not possible that $\hat{i}$ is selected over j by $\mu_{\hat{i}}$, and we have a contradiction. Thus, μ is Pareto-efficient and Claim 2.1 is proved.

Claim 2.2: A rank-partition stable and Pareto-efficient rule is a Near-Boston rule.

Proof: We will show that a Pareto-efficient PRP rule is a Near-Boston rule. Note that this is sufficient to prove the claim since a PRP rule chooses the optimal rank-partition stable matching at each profile $(\succ, P)$, which Pareto-dominates all other rank-partition stable matchings at this profile.

Step A: We show that if $f^{v,p}$ is a Pareto-efficient PRP rule then each school has the coarsest priority partition.

Let $f^{v,p}$ be a Pareto-efficient PRP rule. Suppose that there exists $a \in M$ such that v_a is not the coarsest. For now we assume that $q_a = q_b = q_c = 1$ and thus $n \geq 4$. Specify $(\succ, P)$ as follows. Let $i, j, l, h \in N$ such that

- l has the top-priority for a and j has the lowest priority for a in $\succ_a$. Note that this implies that $l \in v_a^1$ and $j \notin v_a^1$, given our assumption that v_a is not the coarsest.
- i has the lowest priority for a in $\succ_a$, except for j. Thus, $i \succ_a j$.
- j has the top priority for b and i has the lowest priority for b in $\succ_b$.
- h has the top priority for c in $\succ_c$.

All other priorities in $\succ$ are arbitrary.

Case A.1 : $v_a^1 = n - 1$

Let P be given as shown in the table below.

P_i	P_j	P_l	P_h
c	a	c	c
b	b	b	
a		0	

Assume that further preferences for $i, j, l, h \in S$ are arbitrary at P, and for all students $h' \in S \setminus \{i, j, l, h\}$, $P_{h'} \in (0)$ (i.e., they don't have any acceptable school). Assume without loss of generality that $p_i^1 \leq p_l^1 \leq p_j^1$. The rounds of the $f^{v,p}$ procedure at $(\succ, P)$ are displayed below.

Round	a	b	c
1	$\underline{j}$		$i, l, \underline{h}$
2	$\underline{j}$	$i, \underline{l}$	$\underline{h}$
3	$\underline{i}, j$	$\underline{l}$	$\underline{h}$
4	$\underline{i}$	$\underline{j}, l$	$\underline{h}$

In round 1, h is selected over i and j by c, since either h is in a higher priority class than i and l for c, or h wins on the tie-breaker, given that $r_i(l) = r_l(c) = r_h(c) = 1$ and $h \succ_c i, l$.

In round 2, l is selected over i by b, since either l is in a higher priority class than i for b, or l is selected based on preference rank classes, given that $p_i^1 \leq p_l^1$ and $r_i(b) = r_l(b) = 2$, or if neither the priority classes nor the preference classes provide a basis for selecting l over i, then l wins on the tie-breaker, given that $l \succ_b i$.

Note that since $v_a^1 = n - 1$, $i \in v_a^1$ and $i \in v_a^2$, given $\succ_a$. Thus, in round 3 student i is selected over j by a based on the priority rank classes.

In round 4, j is selected over l by b, since either j is in a higher priority class than l for b, or j is selected based on the preference rank classes, given that $p_l^1 \leq p_j^1$ and $r_j(b) = r_l(b) = 2$, or if neither the priority classes nor the preference classes provide a basis for selecting j over l, then j wins on the tie-breaker, given that $j \succ_b l$.

Now note that since i is assigned a and j is assigned b, i and j would prefer to trade their assignments and thus we have a contradiction to Pareto-efficiency.

Case A.2 : $v_a^1 < n - 1$

Let P be given as shown in the table below.

P_i	P_j	P_l	P_h
a	a	c	c
b	b	b	
0		a	

Assume that further preferences for $i, j, l, h \in S$ are arbitrary at P and for all students $h' \in N \setminus \{i, j, l, h\}$, $P_{h'} \in (0)$. Assume without loss of generality that $p_l^1 \leq p_i^1 \leq p_j^1$. The rounds of the $f^{v,p}$ procedure at $(\succ, P)$ are displayed below.

Round	a	b	c
1	$\underline{i}, j$		$l, \underline{h}$
2	$\underline{i}$	$\underline{j}, l$	$\underline{h}$
3	$\underline{i}, l$	$\underline{j}$	$\underline{h}$
4	$\underline{l}$	$i, \underline{j}$	$\underline{h}$

In round 1, i is selected over j by a, since either i is in a higher priority class than j for a, or i wins on the tie-breaker, given that $r_i(a) = r_j(a) = 1$ and $i \succ_a j$. Similarly, h is selected by c over l.

In round 2, j is selected over l by b since j has top priority for b, and thus j may be selected based on the priority classes. If j does not get selected based on the priority classes, then $p_l^1 \leq p_j^1$ and $r_j(b) = r_l(b) = 2$ imply that j may be selected based on preference classes. Finally, if j is not selected based on either the priority or preference classes, then j wins on the tie-breaker, since $j \succ_b l$.

In round 3, l is selected over i by a based on the priority classes, since $l \in v_a^1$ and $i \notin v_a^1$, given that $v_a^1 < n - 1$.

In round 4, j is selected over i by b, since either j is in a higher priority class than i for b, or j is selected based on the preference classes, given that $p_i^1 \leq p_j^1$ and $r_i(b) = r_j(b) = 2$, or if neither the priority classes nor the preference classes provide a basis for selecting j over c, then j wins on the tie-breaker, given that $j \succ_b i$.

Now note that since l is assigned a and j is assigned b, l and j would prefer to trade their assignments and thus we have a contradiction to Pareto-efficiency.

In order to relax the assumption that $q_a = q_b = q_c = 1$ and generalize both Case A.1 and Case A.2, since there exist $a, b, c \in C$ such that $q_a + q_b + q_c < n$, we can introduce additional students with the top priorities for the relevant schools such that each student with the top priority for a relevant school ranks the school first, and this would allow for getting the same contradiction in both cases as before. Since this is a straightforward extension, we omit the tedious details. Finally, note that since v_a is not the coarsest, Cases A.1 and A.2 cover all possible cases and thus Step A is completed.

Step B: We show that if $f^{v,p}$ is a Pareto-efficient PRP rule then it is a Near-Boston rule.

Suppose that $f^{v,p}$ is Pareto-efficient but it is not a Near-Boston rule. Then there exist $j, l \in N$ with at least one preference class each which are minimally size 2. Since the preference classes which are larger than size 1 have to be relevant, that is, there must exist a profile where these sizes matter, it means that there are enough students who can have top priorities at relevant schools, and since we can assume that these students will rank their top-priority schools first, to reduce the technical details we can assume without loss of generality that the size 2 preference classes are the top preference classes for students j and l, and by a similar argument we can assume without loss of generality that $q_a = q_b = q_c = 1$ for schools $a, b, c \in C$, where $q_a + q_b + q_c < n$.

We specify $(\succ, P)$ as follows. Let the preferences for $i, j, l \in S$ be given as shown in the table below.

P_i	P_j	P_l
a	a	b
	b	a

Note that student j has both a and b in the top preference class, and so does student l: $a, b \in P_j^1$ and $a, b \in P_l^1$.

Let $l \succ_a i \succ_a j$ and $j \succ_b l$. Note that all priority partitions are coarse by Step A, since $f^{v,p}$ is a Pareto-efficient rule. The rounds of the PRP rule applied to $(\succ, P)$ are displayed below, with the selections underlined in each round.

Round	a	b
1	$\underline{i}, j$	$\underline{l}$
2	$\underline{i}$	$\underline{j}, l$
3	$i, \underline{l}$	$\underline{j}$

Now note that since l is assigned a and j is assigned b, l and j would prefer to trade their assignments and thus we have a contradiction to Pareto-efficiency. $\square$

Proof of Theorem 3

Theorem 3. *A rank-partition stable rule is strategyproof if and only if it is the Deferred Acceptance rule.*

Proof.

Step 1: *The only strategyproof PRP rule is the DA.*

Let $f^{v,p}$ be a PRP rule which is not the DA. Suppose that $f^{v,p}$ is strategyproof. Since $f^{v,p}$ is not the DA, there exists $c_1 \in C$ such that v_{c_1} is not the finest, and there exists $s^* \in S$ such that p_s^* is not the coarsest. Then there exists rank $r \geq q_{c_1}$, such that the students ranked in the rth and $(r+1)$th positions are in the same priority rank class according to v_{c_1}. Thus, note that $q_{c_1} < n$. Let $s_n \equiv s^*$. Fix $\succ_{c_1}$ with ranks $r_{c_1}(s^*) = r$, $r_{c_1}(s_1) = r + 1$, and for all $i = 2, \ldots, q_{c_1}$, $r_{c_1}(s_i) < r$. Let $c_2, c_3, \ldots, c_m$ be arranged in ascending order of their capacities, that is, $q_{c_2} \leq q_{c_3} \leq \ldots \leq q_{c_m}$. Let $\succ$ be such that for all $t = 2, \ldots, m$, $\succ_{c_t}$ ranks s^* last: $r_{c_t}(s^*) = n$.

Consider each seat of each school as a separate item, denoted by $\bar{c}_1, \ldots, \bar{c}_{\bar{m}}$, where $\bar{m} \equiv \sum_{c \in C} q_c$. For all $t = 1, 2, \ldots, m$, let $Q^t \equiv \sum_{l=1}^{t} q_{c_l}$. Moreover, let $\bar{c}_1, \ldots, \bar{c}_{q_c}$ denote the seats at school c_1, and for all $t = 2, \ldots, m$, let $\bar{c}_{Q^{t-1}+1}, \ldots, \bar{c}_{Q^t}$ denote the seats at school c_t.

- **Case 1:** $n \leq \bar{m}$

  Let $\tilde{m} \geq 2$ be such that $Q^{\tilde{m}} \geq n$ and $Q^{\tilde{m}-1} < n$. Let $P_{s^*} \in (c_2, c_3, \ldots, c_{\tilde{m}-1}, c_1, c_{\tilde{m}})$, and for all $i = 1, \ldots, n-1$, let $P_{s_i} \in (\bar{c}_i)$.

  The rounds of the PRP procedure at $(\succ, P)$ are as follows.

  Round 1: The only conflict is for c_2, among students $s_{q_{c_1}+1}, \ldots, s_{q_{c_1}+q_{c_2}}$ and s^*. Based on $\succ_{c_2}$, since $\succ_{c_2}$ ranks s^* last, either students $s_{q_{c_1}+1}, \ldots, s_{q_{c_1}+q_{c_2}}$ are selected or all students applying to c_2 are selected, depending on v_{c_2}. If it is the latter, then since all students in the applicant pool rank c_2 first, regardless of P the selection is based on the tie-breaker $\succ_{c_2}$. Given that $\succ_{c_2}$ ranks s^* last, students $s_{q_{c_1}+1}, \ldots, s_{q_{c_1}+q_{c_2}}$ are selected by c_2 in this case, too. In sum, s^* is rejected by c_2 and all other applications are accepted in round 1.

  Round 2: s^* applies to c_3, and the only conflict is for c_3, among students $s_{q_{c_1}+q_{c_2}+1}, \ldots, s_{q_{c_1}+q_{c_2}+q_{c_3}}$ and s^*. Here again s^* is rejected by c_3, and all the previous tentative acceptances remain. The arguments are similar to the round 1 arguments, except that since s^* ranks c_3 second, while all other students in the applicant pool for c_3 rank c_3 first, depending on p, s^* could be rejected based on the preference rank classes, if s^* was not rejected already based on the priority rank classes of c_3, given that $\succ_{c_3}$ ranks s^* last. Again, if s^* is not rejected based on priority or preference rank classes by c_3, then s^* is rejected by c_3 based on the tie-breaker $\succ_{c_3}$.

  And so on, we can repeat similar arguments for Rounds 3 to $\tilde{m} - 2:$ s^* applies to schools $c_4, ..., c_{\tilde{m}-1}$ in these rounds, respectively, and gets rejected in each round, while all other students remain tentatively matched to their first-choice school.

  Round $\tilde{m} - 1:$ s^* applies to c_1, and the only conflict is for c_1, among students $s_1, \ldots, s_{q_c}$ and s^*. Since for all $i = 2, \ldots, q_{c_1}$, $r_{c_1}(s_i) < r$, $r_{c_1}(s^*) = r$ and $r_{c_1}(s_1) =$

$r+1$, and since according to v_{c_1} students s^* and s_1, are in the same priority rank class, each student in the applicant pool of c_1 is selected, given $\succ_{c_1}$. Now note that, given $q_{c_2} \leq q_{c_3} \leq \ldots \leq q_{c_m}$ and $Q^{\tilde{m}} \geq n$, s^* is guaranteed to be assigned at worst the $\tilde{m}$th-ranked school under any reported preferences of s^*. Then, since p_{s^*} is not the coarsest, $c_1 \notin P^1_{s^*}$. Since $s_1, \ldots, s_{q_{c_1}}$ rank c_1 first, s^* is rejected by c_1 based on the preference rank classes. This implies that $f^{v,p}_{s^*}(\succ, P) = c_{\tilde{m}}$ and $c_1 P_{s^*} f^{v,p}_{s^*}(\succ, P)$.

Now let $\tilde{P}_{s^*} \in (c_1)$. In round 1 of the PRP procedure at $(\succ, (\tilde{P}_{s^*}, P_{-s^*}))$ based on the priority rank classes specified by v_c, none of the students in the applicant pool for c_1, namely $s_1, \ldots, s_{q_c}$ and s^* are rejected, as shown before. Since all students rank c_1 first in this applicant pool, none of them are rejected based on the preference rank classes. Therefore, the selection is based on the tie-breaker $\succ_{c_1}$. Given that for all $i = 2, \ldots, q_c,\ r_{c_1}(s_i) < r,\ r_{c_1}(s^*) = r$ and $r_{c_1}(s_1) = r+1$, s_1 is rejected and all other applying students are tentatively matched to c_1. Since $P_{s_1} \in (c_1)$, individual rationality of $f^{v,p}$ implies that the procedure stops after round 1 at $(\succ, (\tilde{P}_{s^*}, P_{-s^*}))$, and thus $f^{v,p}_{s^*}(\succ, (\tilde{P}_{s^*}, P_{-s^*})) = c_1$. This means that s^* can manipulate at P via $\tilde{P}_{s^*}$, which contradicts the strategyproofness of $f^{v,p}$.

- **Case 2:** $n > \bar{m}$

  Let $s^* \equiv s_{\bar{m}+1}$ and let $P_{s^*} \in (c_2, c_3, \ldots, c_{m-1}, c_m, c_1)$. For all $i = 1, \ldots, \bar{m}$, let $P_{s_i} \in (\bar{c}_i)$, and for all $i = \bar{m}+2, \ldots, n$, let $P_{s_i} \in (s_i)$. For this case we can use a similar argument as for Case 1, given the above modifications. Since c_1 is ranked last by s^* and P_{s^*} is not the coarsest, $c_1 \notin P^1_{s^*}$. Therefore, we have $f^{v,p}_{s^*}(\succ, P) = 0$ and $c_1 P_s\ f^{v,p}_{s^*}(\succ, P)$. Moreover, we can show similarly to Case 1 that $f^{v,p}_{s^*}(\succ, (\tilde{P}_{s^*}, P_{-s^*})) = c_1$, where $\tilde{P}_{s^*} \in (c_1)$. This means that s^* can manipulate at P via $\tilde{P}_{s^*}$, which contradicts the strategyproofness of $f^{v,p}$.

Step 2: *The only strategyproof rank-partition stable rule is the DA.*[3]

Suppose that φ is a rank-partition stable rule which is strategyproof, and suppose that φ is not the DA. Let φ be rank-partition stable with respect to (v, p). Then, by Step 1, φ is not the PRP rule $f^{v,p}$. Thus, there exists $(\succ, P)$ such that $\varphi(\succ, P) \neq f^{v,p}(\succ, P)$. Then Proposition 1 implies that $f^{v,p}(\succ, P)$ Pareto-dominates $\varphi(\succ, P)$.

[3]Given Step 1, this step also follows from Alva and Manjunath, 2019). We provide a direct proof of this step for completeness.

For all $s \in S$, let $\tilde{P}_s$ be the same as P_s from the first-ranked school to $f^{v,p}_s(P)$, and let all other schools (schools ranked below $f^{v,p}_s(\succ, P)$) be unacceptable to s at preferences $\tilde{P}_s$. In other words, $\tilde{P}_s$ is the truncation of P_s directly below $f^{v,p}_s(\succ, P)$. Let $\bar{P}$ be any preference profile in which for all $s \in S$, $\bar{P}_s \in \{P_s, \tilde{P}_s\}$. Then it easy to see that for all such $\bar{P} \in \mathcal{P}$, $f^{v,p}(\succ, \bar{P}) = f^{v,p}(\succ, P)$.

Let $s_1 \in S$ such that $f^{v,p}_{s_1}(\succ, P) \neq \varphi_{s_1}(\succ, P)$. Then $f^{v,p}_{s_1}(\succ, P)P_{s_1}\varphi_{s_1}(\succ, P)$. Note that $f^{v,p}(\succ, P) = f^{v,p}(\succ, \tilde{P}^1)$, where $\tilde{P}^1 \equiv (\tilde{P}_{s_1}, P_{-s_1})$. Since φ is strategyproof and individually rational, $\varphi_{s_1}(\succ, \tilde{P}^1) = 0$. Now suppose that, for all $s \in S \setminus \{s_1\}$, $f^{v,p}_s(\succ, \tilde{P}^1) = \varphi_s(\succ, \tilde{P}^1)$. Then $\varphi_{s_1}(\succ, \tilde{P}^1) \neq 0$, otherwise φ would be wasteful, which contradicts our assumption that φ is rank-partition stable. Hence, there exists $s_2 \in S \setminus \{s_1\}$ such that $f^{v,p}_{s_2}(\succ, \tilde{P}^1) \neq \varphi_{s_2}(\succ, \tilde{P}^1)$. Then, by Proposition 1, $f^{v,p}_{s_2}(\succ, \tilde{P}^1)P_{s_2}\varphi_{s_2}(\succ, \tilde{P}^2)$. Note that $f^{v,p}(\succ, \tilde{P}^1) = f^{v,p}(\succ, \tilde{P}^2)$, where $\tilde{P}^2 = (\tilde{P}_{s_1}, \tilde{P}_{s_2}, P_{-s_1,s_2})$. Now suppose that for all $s \in S \setminus \{s_1, s_2\}$, $f^{v,p}_s(\succ, \tilde{P}^2) = \varphi_s(\succ, \tilde{P}^2)$. Then
$\varphi_{s_1}(\succ, \tilde{P}^2) \neq 0$ and $\varphi_{s_2}(\succ, \tilde{P}^2) \neq 0$, otherwise φ would be wasteful, which contradicts our assumption that φ is rank-partition stable. Hence, there exists $s_3 \in S \setminus \{s_1, s_2\}$ such that $f^{v,p}_{s_3}(\succ, \tilde{P}^2) \neq \varphi_{s_3}(\succ, \tilde{P}^2)$. And so on, if we keep iterating the same argument then, due to the finiteness of S, we run out of students. This is a contradiction, which shows that if φ is rank-partition stable but not the DA then φ is not strategyproof. $\square$

Proof of Theorem 4

Theorem 4. *Let $f^{v,p}$ be a PRP rule and fix a profile $(\succ, P)$. Let $s \in S$ and $c \in C$ such that $cP_sf^{v,p}_s(\succ, P)$. Let $P'_s \in \mathcal{P}_s$ such that c is in the same or lower preference rank class in P'_s than in P_s, given p. Then $f^{v,p}_s(\succ, (P'_s, P_{-s})) \neq c$.*

Substitutability: If student s is chosen from some applicant pool $S' \subseteq S$ by a school, then student s would still be chosen from the applicant pool T by this school where T is a strict subset of S' and s is a member of T. Formally, a choice function Ch satisfies substitutability if $s \in T \subseteq S' \subseteq S$ and $s \in Ch_c(S')$ then $s \in Ch_c(T)$. It is easy to verify that PRP choice functions satisfy substitutability.

Proof. Let $f^{v,p}$ be a PRP rule and fix a profile $(\succ, P)$. Let $s \in S$, and $c, d \in C$ such that $f^{v,p}_s(\succ, P) = d$ and cP_sd. Let $P'_s \in \mathcal{P}_s$. Let $\bar{\succ}$ denote the constructed priority profile for $((\succ, P), (v, p))$ and let $\bar{\succ}'$ denote the constructed priority profile for $((\succ, (P'_s, P_{-s})), (v, p))$.

Case 1: Assume that P'_s is different from P_s only by reshuffling schools *within* s's preference rank classes only. For the constructed priority profile only the set of schools in each preference class is relevant, while the ordering of the schools within a preference class is irrelevant. Thus, $\bar{\succ} = \bar{\succ}'$. By Proposition 1, $f^{v,p}(\succ, P) = f^{DA}(\bar{\succ}, P)$ and $f^{v,p}(\succ, (P'_s, P_{-s})) = f^{DA}(\bar{\succ}', (P'_s, P_{-s}))$. Then $f^{v,p}(\succ, (P'_s, P_{-s})) = f^{DA}(\bar{\succ}, (P'_s, P_{-s}))$. Therefore, given that $cP_s f^{v,p}_s(\succ, P)$ and thus $cP_s f^{DA}_s(\bar{\succ}, P)$, by the strategyproofness of the DA, we have $f^{DA}\ (\bar{\succ}, (P'_s, P_{-s})) \neq c$. This means that $f^{v,p}(\succ, (P'_s, P_{-s})) \neq c$.
Case 2: Assume that P'_s exchanges two schools (a and b) only when compared to P_s such that a and b are in two *adjacent* preference rank classes. Specifically, assume that a is the bottom-ranked school in its preference rank class, and b is the top-ranked school in the preference rank class just below the one a is in. Otherwise P'_s is the same as P_s, including the preference rankings within each preference rank class.

First note that only the priority rankings of a and b are different in the constructed priority profile $\bar{\succ}'$ compared to $\bar{\succ}$, while the priorities for the remaining schools are the same. Moreover, only the position of student s may change, as follows: s has the same or higher ranking in $\bar{\succ}'_b$ compared to $\bar{\succ}_b$, while s has the same or lower ranking in $\bar{\succ}'_a$ compared to $\bar{\succ}_a$. All other students have the same relative rankings in $\bar{\succ}'_b$ versus $\bar{\succ}_b$, as well as in $\bar{\succ}'_a$ versus $\bar{\succ}_a$.

Claim 4.1. *Let school e be ranked directly above school a by P_s. Then for all $\tilde{e} \in C$ such that $\tilde{e}R_s e$, if $f^{v,p}_s(\succ, P) = \tilde{e}$ then $f^{v,p}_s(\succ, (P'_s, P_{-s})) = \tilde{e}$ and otherwise $f^{v,p}_s(\succ, (P'_s, P_{-s})) \neq \tilde{e}$.*

Proof. Note first that school e is ranked directly above school b by P'_s. Then each round of the PRP procedure in which student s proposes to school $\tilde{e}$ such that $\tilde{e}R_s e$ (and consequently also $\tilde{e}R'_s e$), the PRP rounds at $(\succ, P)$ and at $(\succ, (P'_s, P_{-s}))$ are identical, given that $\tilde{e}P_s a, b$, $\tilde{e}P'_s a, b$ and only Ch_a and Ch_b have changed with respect to the selection of s only, when the two preference profiles are compared. □

Case 2a: $dP_s a$**.** By Claim 10, $f^{v,p}_s(\succ, (P'_s, P_{-s})) = f^{v,p}_s(\succ, P) = d$.

Case 2b: $d = a$**.** Given Claim 10, since $cP_s d$, $f^{v,p}_s(\succ, (P'_s, P_{-s})) \neq c$. Claim 10 also implies that we have one of the following scenarios:

scenario (i): $f^{v,p}_s(\succ, (P'_s, P_{-s})) = d = a$

scenario (ii) : $f^{v,p}_s(\succ, (P'_s, P_{-s})) = b$ (note: $b \neq c$, since $cP_s dP_s b$)

scenario (iii): $dP_s f_s^{v,p}(\succ, (P'_s, P_{-s}))$

Case 2c: $d = b$**.** Given Claim 10, for all $\tilde{e} \in C$ such that $\tilde{e} R_s e$, $f_s^{v,p}(P'_s, P_{-s}) \neq \tilde{e}$. Suppose that s is rejected by school $d = b$ in the PRP procedure at profile $(\succ, (P'_s, P_{-s}))$. Then, since the PRP rule is not wasteful, there exists at least one student $\hat{s} \in S$ such that $f_{\hat{s}}^{v,p}(\succ, (P'_s, P_{-s})) = b$ and $f_{\hat{s}}^{v,p}(P) \neq b$. Since $f_{\hat{s}}^{v,p}(\succ, (P'_s, P_{-s})) = b$ and $bP_s f_s^{v,p}(\succ\ (P'_s, P_{-s}))$, given that the PRP rule $f^{v,p}$ is stable with respect to the constructed priority profile at each preference profile by Proposition 1, $\hat{s} \bar{\succ}'_b s$. Since s cannot have a lower position in $\bar{\succ}'_b$ compared to $\bar{\succ}_b$, while all other students' relative positions are unchanged, this implies that $\hat{s} \bar{\succ}_b s$. If $bP_{\hat{s}} f_{\hat{s}}^{v,p}(\succ, P)$, then since $\hat{s} \bar{\succ}_b s$ and $f(\succ, P)$ is stable with respect to $\bar{\succ}$, we have a contradiction. Thus, since $f_{\hat{s}}^{v,p}(\succ, P) \neq b$, we have $f_{\hat{s}}^{v,p}(\succ, P) P_{\hat{s}} b$. Note that this implies that $\hat{s}$ does not apply to school b in the $f^{v,p}$ procedure at P.

Assume that s is rejected by e in round $k-1$ of the $f^{v,p}$ procedure applied to both $(\succ, P)$ and $(\succ, (P'_s, P_{-s}))$. This is without loss of generality due to Claim 10. Then s applies to school a in Step k at $(\succ, P)$, and s applies to school b in round k at $(\succ, (P'_s, P_{-s}))$. Let $T \subset S$ be the set of students who apply to b in any round after round $k-1$ in the $f^{v,p}$ procedure at $(\succ, P)$, and let $T' \subset S$ be the set of students who apply to b in any round after round $k-1$ in the $f^{v,p}$ procedure at $(\succ, (P'_s, P_{-s}))$. We will show that $T' \subseteq T$.

It follows from Claim 10 that in round k at $(\succ, P)$ each student except s applies to the same school as in round k at $(\succ, (P'_s, P_{-s}))$. If student s gets rejected by school a at $(\succ, P)$ in round k then there is no difference in applicants to b at the two profiles after round k and $T' = T$, due to the substitutability property of the school choice functions. If student s gets accepted and some other student gets rejected by a at $(\succ, P)$ who is not rejected by a at $(\succ, (P'_s, P_{-s}))$ then this student applies to her next most-preferred school, which in turn my cause another student to be rejected by a school at $(\succ, P)$ who is not rejected at $(\succ, (P'_s, P_{-s}))$, etc., creating a rejection chain at $(\succ, P)$. Therefore, all students who applied to b at $(\succ, (P'_s, P_{-s}))$, and potentially more students, apply to b at $(\succ, P)$ after round k, since all students involved in this rejection chain get a lower-ranked school at $(\succ, P)$ than at $(\succ, (P'_s, P_{-s}))$, given the substitutability property of the school choice functions. Thus, if $t \in T'$ then $t \in T$, as claimed. Note, however, that this contradicts the existence of student $\hat{s}$ since, as shown above, $\hat{s} \in T' \setminus T$.

Case 2d: bP_sd**.** Note that $b \neq c$, since c is either in the same or a lower preference rank class in P'_s than in P_s, given the fixed p_s. We will show that one of the following three scenarios holds:

scenario (i): $f_s^{v,p}(\succ, (P'_s, P_{-s})) = d$;

scenario (ii): $f_s^{v,p}(\succ, (P'_s, P_{-s})) = b$;

scenario (iii): $dP_sf_s^{v,p}(\succ, (P'_s, P_{-s}))$.

By Claim 4.1, for all $\tilde{e} \in C$ such that $\tilde{e}R_se$, $f_s^{v,p}(\succ, (P'_s, P_{-s})) \neq \tilde{e}$. If $f_s^{v,p}(\succ, (P'_s, P_{-s})) = b$ then we are done, as scenario (ii) holds in this case. Thus, we can assume that $f_s^{v,p}(\succ, (P'_s, P_{-s})) \neq b$. Then $bP_sf_s^{v,p}(\succ, (P'_s, P_{-s}))$.

First we show that $f_s^{v,p}(\succ, (P'_s, P_{-s})) \neq a$. Suppose that $f_s^{v,p}(\succ, (P'_s, P_{-s})) = a$. Then we can show that $f_s^{v,p}((\succ, P)) = a$, using a similar argument to the one for Case 2c. For Case 2c we showed that if $f_s^{v,p}(\succ, P) = b$ then $f_s^{v,p}(\succ, (P'_s, P_{-s})) = b$, and here we can use a similar argument applied to a instead of b, to show that $f_s^{v,p}(\succ, P) = a$, which contradicts $f_s^{v,p}(\succ, P) = d$, since aP_sbP_sd. Therefore, $f_s^{v,p}(\succ, (P'_s, P_{-s})) \neq a$, and thus we have shown so far that $aP'_sf_s^{v,p}(\succ, (P'_s, P_{-s}))$.

Claim 4.2. *Assume that P'_s is different from P_s only by the ordering of schools within s's preference rank class which contains the school assigned to s at $(\succ, P)$, say school c, that is, when $f_s^{v,p}(\succ, P) = c$. Assume furthermore that the upper contour set of school c in this preference class is weakly smaller at P'_s than at P_s, that is, if c' is in the same preference class in P_s as c, and if $c'P'_sc$ then $c'P_sc$. Then $f_s^{v,p}(\succ, (P'_s, P_{-s})) = c$.*

Proof. As noted in the argument for Case 1, for the constructed priority profile only the set of schools in each preference class is relevant, while the ordering of the schools within a preference class is irrelevant. Thus $\bar{\succ} = \bar{\succ}'$. Then $f^{v,p}(\succ, P) = f^{DA}(\bar{\succ}, P)$ and $f^{v,p}(\succ, (P'_s, P_{-s})) = f^{DA}(\bar{\succ}', (\succ, (P'_s, P_{-s}))) = f^{DA}(\bar{\succ}, (P'_s, P_{-s}))$. Therefore, if $f_s^{v,p}(\succ, P) = c$, or equivalently $f_s^{DA}(\bar{\succ}, P) = c$, by strategyproofness of the DA, $f_s^{DA}(\bar{\succ}, (P'_s, P_{-s})) = c$ or, equivalently, $f_s^{v,p}(\succ, (P'_s, P_{-s})) = c$. □

Given Claim 4.2, we show next that if $f_s^{v,p}(\succ, (P'_s, P_{-s})) \neq b$ then dR_s $f_s^{v,p}(\succ, (P'_s, P_{-s}))$. Since $a, bP_sf_s^{v,p}(\succ, P)$, $a, bP'_sf_s^{v,p}(\succ, (P'_s, P_{-s}))$, and only the order of a and b has changed between P_s and P'_s, where a and b are adjacent in the preference ordering, suppose that there exists $c \in C$ such that $bP_s cP_s d$ and $f_s^{v,p}(\succ, (P'_s, P_{-s})) = c$. This means that $aP'_s cP'_s d$.

Consider P''_s which is the same as P'_s, except that c is lifted to be ranked directly below a (note: if c was already ranked directly below a by P'_s, then $P'_s = P''_s$). Since this change only affects the preference class that contains a, c and d, while all other orderings in P'_s are preserved, Claim 4.2 implies that $f_s^{v,p}(\succ, (P''_s, P_{-s})) = c$. Consider $\hat{P}_s$ which is the same as P_s, except that c is lifted to the top of the preference class that contains b, c and d, while all other orderings in P_s are preserved.

Now consider P''_s versus $\hat{P}_s$. Given that $f_s^{v,p}(\succ, (P''_s, P_{-s})) = c$, Claim 4.1 implies that for all $\tilde{e}R''_s e$, $f_s^{v,p}(\succ, (\hat{P}_s, P_{-s})) \neq \tilde{e}$. Now suppose that $f_s^{v,p}(\succ, (\hat{P}_s, P_{-s})) \neq a$. We will show that then $f_s^{v,p}(\succ, (P''_s, P_{-s})) = c$ implies that $f_s^{v,p}(\succ, (\hat{P}_s, P_{-s})) = c$.

Suppose that s is rejected by school c in the $f^{v,p}$ procedure at profile $(\succ, (\hat{P}_s, P_{-s}))$. Then, since the PRP rule is not wasteful, there exists at least one student $\tilde{s} \in S$ such that $f_{\tilde{s}}^{v,p}(\succ, (\hat{P}_s, P_{-s})) = c$ and $f_{\tilde{s}}^{v,p}(\succ, (P''_s, P_-s)) \neq c$. Since $f_{\tilde{s}}^{v,p}(\succ, (\hat{P}_s, P_{-s})) = c$ and $cP_s f_s^{v,p}(\succ, (\hat{P}_s, P_{-s}))$, given that the PRP rule $f^{v,p}$ is stable with respect to the constructed priority profile at each preference profile by Proposition 1, $\tilde{s}\hat{\succ}_c s$, where $\hat{\succ}$ is the constructed priority profile at $(\succ, (\hat{P}_s, P_{-s}))$. Thus, since c is in the same preference class in P''_s and in $\hat{P}_s$, we also have $\tilde{s} \succ''_c s$, where $\succ''$ is the constructed priority profile at $(\succ, (P''_s, P_{-s}))$. If $cP_{\tilde{s}} f_{\tilde{s}}^{v,p}(\succ, (P''_s, P_{-s}))$, then since $\tilde{s} \succ''_c s$ and $f(\succ, (P''_s, P_{-s}))$ is stable with respect to $\succ''$, we have a contradiction. Thus, $f_{\tilde{s}}^{v,p}(\succ, (P''_s, P_-s))P_{\tilde{s}}c$, which implies that $\tilde{s}$ does not apply to school c in the $f^{v,p}$ procedure at $(\succ, (P''_s, P_-s))$.

Assume that s is rejected by e in round $k-1$ of the $f^{v,p}$ procedure applied to $(\succ, (P''_s, P_{-s}))$ and to $(\succ, (\hat{P}_s, P_{-s}))$. Then s applies to school b in round k at $(\succ, (P''_s, P_{-s}))$ and to school a at $(\succ, (\hat{P}_s, P_{-s}))$. Let $V \subset S$ be the set of students who apply to c in any round after round $k-1$ in the $f^{v,p}$ procedure at $(\succ, (P''_s, P_{-s}))$ and let $V' \subset S$ be the set of students who apply to c in any round after round $k-1$ in the $f^{v,p}$ procedure at $(\succ, (\hat{P}_s, P_{-s}))$. We will show that $V' \subseteq V$.

It follows from Claim 4.1 that in round k at $(\succ, (P''_s, P_{-s}))$ each student except

s applies to the same school as in round k at $(\succ, (\hat{P}_s, P_{-s}))$. If student s gets rejected by school b at $(\succ, (P''_s, P_{-s}))$ in round k then there is no difference in applicants to a at the two profiles after round k, which implies that $V' = V$, due to the substitutability property of the school choice functions. If student s gets accepted by b at $(\succ, (P''_s, P_{-s}))$ and some other student gets rejected after round $k-1$ who is not rejected by b at $(\succ, (\hat{P}_s, P_{-s}))$, then this student applies to her next most-preferred school, which in turn my cause another student to be rejected by a school at $(\succ, (P''_s, P_{-s}))$ who is not rejected at $(\succ, (\hat{P}_s, P_{-s}))$, etc., creating a rejection chain at $(\succ, (P''_s, P_{-s}))$. Therefore, all students who applied to c at $(\succ, (\hat{P}_s, P_{-s}))$, and potentially more students, apply to c at $(\succ, (P''_s, P_{-s}))$, since all students involved in this rejection chain get a lower-ranked school at $(\succ, (P''_s, P_{-s}))$ than at $(\succ, (\hat{P}_s, P_{-s}))$, again due to the substitutability property of the school choice functions. Thus, if $v \in V'$ then $v \in V$, as claimed. This, however, contradicts the existence of student $\tilde{s}$ since, as shown above, $\tilde{s} \in V' \setminus V$. Therefore, $f^{v,p}_s(\succ, (\hat{P}_s, P_{-s})) \in \{a, c\}$. Finally note that, given P_s, Claim 4.2 implies that $f^{v,p}_s(\succ, (\hat{P}_s, P_{-s})) = d$. This is a contradiction. Therefore,
$f^{v,p}_s(\succ, (P'_s, P_{-s})) \neq c$ and it follows that either $f^{v,p}_s(\succ, (P'_s, P_{-s})) = d$ or
$dP_s f^{v,p}_s(\succ, (P'_s, P_{-s}))$, corresponding to scenarios (i) and (iii) respectively.

Conclusion of the proof:

By repeatedly applying the transformation of P_s in Case 1, which allows for the reshuffling of schools within the preference rank classes of students, as well as the transformation of P_s in Case 2, which allows for exchanging two adjacent schools in the ordering which switches the preference rank classes in which these two schools are, we can transform P_s into an arbitrary $\check{P}_s$ such that c is never moved to a higher preference rank class in any round of the transformation. Thus, based on the proofs for the individual cases above, we can conclude that one of the following three cases holds for $f^{v,p}_s(\succ, (P'_s, P_{-s}))$ for each intermediate round P'_s when transforming P_s into an arbitrary $\check{P}_s$:

1. $f^{v,p}_s(\succ, (P'_s, P_{-s}))$ is the same as in the previous round, denoted by d, and thus does not equal c in Case 1, Case 2a, Case 2b scenario (i), Case 2c, and Case 2d scenario (i).

2. $f_s^{v,p}(\succ, (P'_s, P_{-s}))$ becomes the school which moves up to a higher preference rank class, denoted by b, where $b \neq c$, since c never moves to a higher preference rank class in any round of the transformation. This happens in Case 2b scenario (ii), and Case 2d scenario (ii).

3. $f_s^{v,p}(\succ, (P'_s, P_{-s}))$ is less preferred than the school assigned in the previous outcome which is denoted by d: $dP_s f_s^{v,p}(\succ, (P'_s, P_{-s}))$, where $cP_s d$. Thus, $f_s^{v,p}(\succ, P'_s, P_{-s}) \neq c$. This happens in Case 2b scenario (iii) and in Case 2d scenario (iii).

In sum, for arbitrary $\check{P}_s$ such that c is in the same or lower preference rank class in $\check{P}_s$ than in P_s, $f_s^{v,p}(\succ, (\check{P}_s, P_{-s})) \neq c$. $\square$