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Preference Aggregation for Couples*

Rouzbeh Ghouchani† Szilvia Pápai‡

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Abstract

We study the aggregation of a couple's preferences over their respective jobs when they enter a centralized labor market jointly, such as the market for assigning hospital residencies to medical students. Usually in such markets couples need to submit joint preferences over pairs of jobs. Starting from two individual preference orderings over jobs, we first study the Lexicographic and the Rank-Based Leximin aggregation rules, and then propose a family of aggregation rules, the k-Lexi-Pairing rules, and provide an axiomatic characterization of these rules. The parameter k indicates the degree of selfishness for one partner (and altruism for the other partner), with the least selfish Rank-Based Leximin rule at one extreme and the most selfish Lexicographic rule at the other extreme. Since couples care about geographic proximity, we also identify a simple parametric family of preference aggregation rules which build on the k-Lexi-Pairing rules and reflect the couple's preference for togetherness.

Keywords: matching with couples; preference aggregation; leximin; compromise; togetherness

JEL Classification Codes: D71, D47

1 Introduction

Finding jobs has become more complicated due to the increase in the number of couples who are interested in entering the labor market together. Since there used to be many more men than women seeking positions, only a few couples would apply for jobs simultaneously. This is no longer the case today, which has considerable implications for centralized entry-level labor markets such as the medical residency matches in the US, the UK, and Canada. The National Resident Matching Program (NRMP) in the US, the most prominent example of such a labor market, assigns thousands of medical school graduates to hospital intern positions through a

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central clearing house each year (Roth 1984, Roth and Peranson 1999). As discussed by Roth and Peranson (1999) and Klaus et al. (2007), among others, if the centralized matching procedure does not accommodate couples' wishes to find jobs in the same geographical area, then they might prefer to apply for positions directly, instead of going through the centralized matching system which may easily assign them jobs that are far away from each other, leaving the couple unhappy.

Beyond the workings of the matching algorithm itself, a more fundamental issue is that a stable matching may not even exist (Roth 1984, Cantala 2004). Klaus and Klijn (2005) demonstrate that if both couples and hospitals have responsive preferences then there is always a responsive preference extension for which a stable matching exists, and show that responsive preferences constitute a maximal preference domain in this sense to guarantee the existence of a stable matching (see also Klaus et al. 2009). Responsiveness means that unilateral improvements according to the preference of one partner are beneficial for the couple. Khare et al. (2018) characterize the responsive preferences of couples under which a stable matching always exists. Even under the milder requirements of Klaus and Klijn (2005), existence of a stable matching can only be achieved when couples' preferences are responsive, and responsiveness essentially reduces the couple's joint preferences to two independent individual preferences. This rules out the complementary preferences of couples over jobs that arise due to distance considerations, the most salient characteristic of couples' preferences, which is not surprising since some substitutability condition is typically required for the existence of a stable matching.

Dutta and Massó (1997) studies externalities in preferences and provides possibility results for couples under specific preference restrictions. An alternative approach of relaxing the stability requirement is presented by Jiang and Tian (2014). Khare and Roy (2018) pursues further the issue of the existence of stable matchings in markets with couples when preferences are not responsive. Delacrétaz (2019) and Sidibé (2020) are other recent studies that are relevant for the couples' matching problem, as they study matching with agents of different sizes (i.e., agents may require multiple items on the other side of the market). Even if a stable matching with couples exists at a given preference profile, it is not guaranteed that an algorithm will be able to identify and choose a stable matching at such a preference profile. Klaus et al. (2007) show that the new NRMP algorithm (see also Roth and Peranson (1999)) may not reach an existing stable matching, even when couples' preferences are responsive. They also demonstrate that the new NRMP algorithm may be manipulable by couples acting as singles.

Centralized labor markets with explicitly recognized couples, such as the NRMP and the Canadian Resident Matching Service (CaRMS) today, require participating couples to report their joint preference orderings over pairs of positions, and the relevant matching theory literature takes these joint preferences to be exogenously given. However, an overlooked issue is that given the two partners' respective preferences over individual positions, it is not necessarily clear what the couple's joint preferences are. It is natural for each partner to be aware of their individual preference ordering over the jobs, but it is not obvious that couples know or understand well their preferences over pairs of positions which reflect the preferences of both partners even without geographical considerations, and especially when it comes to incorporating the complementary nature of the two positions.

The preference aggregation issue for a couple has not been addressed by the social choice theory literature either, which mainly focuses on aggregating preferences over social outcomes. Our problem formally differs from the preference aggregation literature in several aspects. The aggregation for couples has two preference rankings over individual positions as input, and a joint preference ranking over pairs of positions as an output. Standard preference aggregation rules going back to Arrow (1963) take identical inputs and turn them into an output of the same form as the inputs. In the preference aggregation literature social outcomes are typically public in nature and individuals care about all aspects of the outcome. Some work has been done on economic domains assuming selfishness (i.e., individuals care about their own allocation only) which are surveyed in Bossert and Weymark (2008) and Le Breton and Weymark (2011). Bordes and Le Breton (1990) study Arrow consistency in various matching models. Kalai and Ritz's setup (1980) comes closest to ours, since they study the same preference aggregation model as us, but with n agents. Unlike us, they focus only on Arrow social welfare functions. Clearly, our model also differs from usual models in that we only have two agents, while preference aggregation is typically considered for an arbitrary number of agents. This makes our task simpler, but it also renders tie-breaking more important since ties arise frequently with two agents only. Finally, and importantly, in the labor market context we need to worry about complements in couples' preferences due to geographical considerations, which is absent from the preference aggregation literature and only pertains to aggregating preferences over private assignments with specific restricted externalities arising from a preference for compatibility of the private assignments.

In this paper we study how to form a joint preference ordering by aggregating a couple's respective individual preferences over single jobs. Can we find consistent, efficient, and fair methods to aggregate the two individual preferences? This aggregation is of interest since reaching a consensus, a feasible compromise that reflects the couple's preferences, may be difficult. We believe, furthermore, that a clear preference aggregation method is not only of relevance to couples when submitting their joint preferences, but it could also play a role in the matching procedure itself, if the matching algorithm is modified accordingly. If couples were restricted to submitting joint preferences with a simplified structure, captured by a handful of parameters only which still allow couples the freedom to express their joint preferences, it may become possible to design more effective matching mechanisms, which would take advantage of the clear structure of the preferences submitted by couples. In addition to being able to construct better matching algorithms, a parametric family of a couple's preferences may also help with the evaluation of the performance of the matching algorithm as a function of different parameter values submitted by participating couples. In light of the severe difficulties with stability and incentives in the presence of couples, as demonstrated by the extant literature, our hope is that such an approach may turn out to be useful. Thus, one of the contributions of this paper is that it initiates this new approach to matching with couples, in addition to proposing and analyzing specific families of preference aggregation rules for couples.

We start by considering two interesting pairing rules in this setting, the Lexicographic (serially dictatorial) and the Rank-Based Leximin rules. In our terminology lexicographic means that preferences are lexicographic in the way they prioritize the two partners, as opposed to lexicographically considering different aspects of the respective individual rankings of jobs. In the absence of cardinal utilities which would be difficult to elicit, and if elicited would further escalate the incentive problems for the matching rule, our aggregation rules are based on the ordinal rankings of individual positions by the two partners and rely on a comparison of these rank numbers between them. Comparing rank numbers and differences in rank numbers may be unusual, given that the elicited preferences are ordinal in nature, but such comparisons based on rank numbers are inevitable in this setting if we don't want to restrict ourselves to serial dictatorships exclusively, hence the name "Rank-Based" Leximin rule.

Both aggregation rules (or pairing rules, as we refer to them) are characterized by appealing normative axioms. The characterization of the Lexicographic rule is a classic one and follows from Kalai and Ritz (1980), while the Rank-Based Leximin rule is characterized by a new set of axioms in our setting (Theorem 1). We then identify a class of pairing rules, the General Lexi-Pairing rules, which includes both the Lexicographic and the Rank-Based Leximin rules, and argue that it is desirable to further narrow down this set of rules when studying couples' preference aggregation choices and their incentives in labor markets, in order to restrict the couple to report a member of a simple but flexible family of preferences. We propose a family of aggregation rules parameterized by k, the k-Lexi-Pairing rules, which yields a ranking of paired positions for a couple prior to taking into account the complementarities in preferences. The parameter k represents the degree to which one partner is favored, which indicates the degree of "selfishness" for this partner, where the least selfish leximin preference aggregation is at one extreme, and the most selfish lexicographic preference aggregation is at the other extreme. Symmetrically, k also shows the extent of "altruism" of the other partner, with the least altruistic rule being the leximin preference aggregation, and the most altruistic the lexicographic aggregation. Given that the aggregation problem is symmetric in the two partners, we assume that the partners are interchangeable and simplify our exposition by omitting the symmetric case where partner 1 is altruistic and partner 2 is selfish.

We provide an axiomatic characterization of k-Lexi-Pairing rules (Theorem 2), which shows that these are the only rules satisfying a natural efficiency requirement (Strong Pareto) and an axiom requiring a uniform degree of concessions that determines when to take into account the partner's preferences ahead of one's own preferences (k-Compromise), together with a consistent tie-breaking axiom (k-Threshold-Consistency). Furthermore, we introduce a general framework for considering geographic location and proximity, and propose a modification of Lexi-Pairing rules which is based on a togetherness parameter that we introduce, which allows the couple to incorporate their subjective preferences for the proximity of their respective jobs. Preferences for proximity have been considered by Dutta and Massó (1997) and by Khare and Roy (2018) in the simple form of preference for getting positions at the same firm or hospital, while Cantala (2004) and Sethuraman et al. (2018) offer more general but still limited geographic considerations. We introduce two natural axioms for modifying the joint preferences of the couple to reflect their preference for proximity, which are satisfied by our proposed aggregation rules for couples, the Couple-Lexi-Pairing rules, and find that the characterization result of Theorem 2 still holds in essence when we modify the axioms to take into account the togetherness parameter.

In the formal exposition we use the terminology of medical residency matching and call the two partners in a couple doctors, while the jobs are referred to as hospitals. Nonetheless, the analysis is relevant for couples in any centralized labor market which involves matching not only single individuals but also couples to jobs. Moreover, all the results pertain to a general two-agent preference aggregation problem over private alternatives, except for the findings in Section 7 which focuses on geographic considerations, and thus it is specific to couples in labor markets.

2 Setup

There is a set of q hospitals H, and two doctors denoted by $i \in \{1, 2\}$. We assume that each hospital has at least two positions, where each position of a hospital $h \in H$ is assumed to be identical. Thus, we regard $H \times H$ as the set of of paired hospital positions, given that positions at each hospital are the same, and $(h_1, h_2) \in H \times H$ indicates a pair of positions where h_1 denotes the position for doctor 1 and h_2 denotes the position for doctor 2. Note that if $h_1 = h$ and $h_2 = h$ for some $h \in H$ then both doctors are matched to a position at the same hospital. Although we refer to "hospital pairs" throughout the paper for simplicity, it is understood that a hospital pair may consist of two positions at the same hospital.

Each doctor has a strict preference ordering over the set of hospitals H, indicating the doctor's individual preferences over the hospitals. The individual preference ordering of doctor $i \in \{1, 2\}$ is expressed by the ranking of each hospital h_i, denoted by $r_i(h_i) \in \{1, \dots, q\}$, where $r_i(h_i) < r_i(h'_i)$ means that doctor i prefers hospital h_i to h'_i, since h_i has a lower rank number than h'_i. We assume that each hospital position is acceptable to both doctors, since each doctor would rather get a job than remain unmatched. Let R denote the set of individual hospital rankings. For each doctor $i \in \{1, 2\}$, let $r_i \in R$ denote a particular ranking of all hospitals in H by doctor i. Let $\mathcal{P}$ denote the set of strict preference orderings over the ordered pairs of hospitals, that is, the set of aggregated preference orderings of paired hospital positions. Then $P \in \mathcal{P}$ is a strict preference ordering over $H \times H$ and represents the joint preferences of the two doctors.

A **pairing rule** is a preference aggregation function for two doctors, which maps from two strict individual preference orderings of individual hospital positions to one strict preference ordering of paired hospital positions. Formally, a pairing rule is a function $\varphi : R \times R \to \mathcal{P}$, specifying the preference aggregation of the respective individual hospital rankings of the two doctors. We will also use the notation $(h_1, h_2)P(h'_1, h'_2)$ to indicate that (h_1, h_2) is preferred to (h'_1, h'_2) in the joint preferences $P \in \mathcal{P}$.

3 The Lexicographic Rule: No Interpersonal Comparisons

We begin with the Strong Pareto axiom, which would naturally be satisfied by the aggregated preferences when there are no complementary preferences over hospital positions. As usual, (h'_1, h'_2) **Pareto-dominates** (h_1, h_2) if $r_1(h_1) \geqslant r_1(h'_1)$ and $r_2(h_2) \geqslant r_2(h'_2)$, with at least one strict inequality.

Strong Pareto. P satisfies Strong Pareto at (r_1, r_2) if, for all (h_1, h_2) and (h'_1, h'_2) such that (h'_1, h'_2) Pareto-dominates (h_1, h_2), $(h'_1, h'_2)P(h_1, h_2)$. A pairing rule φ satisfies Strong Pareto if for all $(r_1, r_2) \in R \times R$, $\varphi(r_1, r_2)$ satisfies Strong Pareto.

Strong Pareto is closely related to the responsiveness notions for doctors used by Klaus and Klijn (2005) and Khare et al. (2018), although their models are slightly different from ours. We consider Strong Pareto our most basic axiom, since it simply says that whenever there is an agreement between the two partners over two hospital pairs, the aggregation honors their common preferences. The question is how to rank two hospital pairs when the partners disagree over their rankings based on their individual hospital matches, and some of the other axioms directly address this case.

Given that we only have ordinal preferences as input, which is consistent with the matching theory literature, imposing an independence of irrelevant alternatives (IIA) axiom may not necessarily be deemed too restrictive, even though it implies the lack of interpersonal comparisons. When IIA is combined with Strong Pareto, we immediately get Arrow's impossibility result since only a serial dictatorship rule satisfies both axioms, given that the aggregate preferences need to be transitive. We refer to serial dictatorships as lexicographic rules, since they prioritize one partner over the other in a lexicographic manner. Klaus and Klijn (2005) refers to the same as the leader-follower responsive preferences.

Independence of Irrelevant Alternatives (IIA). A pairing rule φ satisfies Independence of Irrelevant Alternatives if the following holds for all (h_1, h_2), $(h_1', h_2') \in H \times H$ and (r_1, r_2), $(\bar{r}_1, \bar{r}_2) \in R \times R$. If

(i) $r_1(h_1) < r_1(h_1')$ if and only if $\bar{r}_1(h_1) < \bar{r}_1(h_1')$ and

(ii) $r_2(h_2) < r_2(h_2')$ if and only if $\bar{r}_2(h_2) < \bar{r}_2(h_2')$,

then $(h_1', h_2')P(h_1, h_2)$ if and only if $(h_1', h_2')\bar{P}(h_1, h_2)$, where P denotes $\varphi(r_1, r_2)$ and $\bar{P}$ denotes $\varphi(\bar{r}_1, \bar{r}_2)$.

In the following definition of the Lexicographic rule, we assume without loss of generality that doctor 1 is the first dictator (or the leader), that is, doctor 1 is the partner whose preferences always dominate the other's.

Lexicographic rule

Fix $(r_1, r_2) \in R \times R$ and let P denote the paired preference ordering $\varphi(r_1, r_2)$, where φ is the Lexicographic rule. Then for $(h_1, h_2), (h_1', h_2') \in H \times H$, $(h_1', h_2')P(h_1, h_2)$ if one of the following two cases holds:

1. $r_1(h_1) > r_1(h_1')$;

2. $r_1(h_1) = r_1(h_1')$ and $r_2(h_2) > r_2(h_2')$.

This is the Lexicographic rule favoring doctor 1. If doctor 1 prefers h_1' to h_1, or if $h_1' = h_1$ and doctor 2 prefers h_2' to h_2 then the pair (h_1', h_2') is preferred to the pair (h_1, h_2) according to the joint preferences.

Proposition 1. (Characterization of the Lexicographic rule)

A pairing rules satisfies Strong Pareto and IIA if and only if it is the Lexicographic rule.

This is Arrow's famous impossibility theorem adopted to our setting. We omit the straightforward proof of the proposition which also follows from Kalai and Ritz (1980).

Arguably, in our setting a serial dictatorship is not as undesirable as in other contexts. First of all we have private outcomes, so if agents are selfish and only care about their own allocations then having a consensus over ranking two hospital pairs is more likely than in the public outcome setting. Secondly, we only have two agents, and thus favoring one over the other is not nearly as extreme as favoring one agent over all other agents when the number of agents is large. Thirdly, in the couple preference aggregation problem specifically, it may be desirable for the two spouses to favor one of them over the other, since the two partners wish to cooperate with each other and there may be consensus that one partner's preferences should "weigh" more than the other's due to various reasons. For example, if one is likely to face a tougher job market than the other, or if one spouse has a bigger need for a good job placement than the other for any reason, then the other partner may consent to favoring this partner's preferences.

Nonetheless, the Lexicographic rule is an extreme asymmetric pairing rule, and we want to study other pairing rules that treat agents more symmetrically. For this we need to assume that we can make some interpersonal comparisons, which will amount to treating the ordinal rankings of the agents as comparable utility levels, since we do not have information about preference intensities.[1] In the next section we consider axioms that ask for interpersonal comparisons based on the respective preference ranks and aim to ensure some degree of equity between the two partners, together with some utilitarian notions of efficiency.

4 Equity and the Rank-Based Leximin Rule

We start by presenting some desirable fairness and efficiency properties of pairing rules based on the comparisons of preference rank numbers. All the axioms from here on are defined for a specific pair of individual rankings (r_1, r_2), and a pairing rule φ satisfies an axiom if for all $(r_1, r_2) \in R \times R$, the paired ranking $\varphi(r_1, r_2)$ satisfies the axiom at (r_1, r_2).

First we introduce a few preliminary notions and terminology. Given $(r_1, r_2) \in R \times R$, (h'_1, h'_2) **cross-dominates** (h_1, h_2) if $r_1(h_1) \geqslant r_2(h'_2)$ and $r_2(h_2) \geqslant r_1(h'_1)$, with at least one inequality, but (h'_1, h'_2) does not Pareto-dominate (h_1, h_2). Moreover, (h'_1, h'_2) **dominates** (h_1, h_2) if (h'_1, h'_2) either Pareto-dominates or cross-dominates (h_1, h_2). We will also say that there is no dominance relation between two hospital pairs if neither dominates the other. For a given (h_1, h_2), let $\Sigma = r_1(h_1) + r_2(h_2)$ be the **sum** of the two rankings, and let $g = |r_1(h_1) - r_2(h_2)|$ be the **gap** between the two rankings, i.e., the absolute value of the difference between the two rankings. In the following, let $\Sigma, \Sigma', \tilde{\Sigma}, \ldots$ denote the sum of $(h_1, h_2), (h'_1, h'_2), (\tilde{h}_1, \tilde{h}_2), \ldots$ respectively. Similarly, let $g, g', \tilde{g}, \ldots$ denote the corresponding gaps.

Cross-Dominance. P satisfies Cross-Dominance at (r_1, r_2) if, for all (h_1, h_2) and (h'_1, h'_2) such that (h'_1, h'_2) cross-dominates (h_1, h_2), $(h'_1, h'_2) P (h_1, h_2)$.

Dominance. P satisfies Dominance at (r_1, r_2) if, for all (h_1, h_2) and (h'_1, h'_2) such that (h'_1, h'_2) dominates (h_1, h_2), $(h'_1, h'_2) P (h_1, h_2)$.

Note that Dominance is equivalent to the conjunction of Strong Pareto and Cross-Dominance. Dominance is also known as Suppes-Sen dominance in more general settings.

For $(h_1, h_2) \in H \times H$, let

$$\text{Max} \equiv \max\,(r_1(h_1), r_2(h_2));$$

[1]Bossert and Weymark (2004) provide a comprehensive treatment of the literature on social choice with interpersonal utility comparisons.

$$\text{Min} \equiv \min\ (r_1(h_1), r_2(h_2)).$$

For $(h_1', h_2') \in H \times H$, let

$$\text{Max}' \equiv \max\ (r_1(h_1'), r_2(h_2'));$$
$$\text{Min}' \equiv \min\ (r_1(h_1'), r_2(h_2')).$$

With this notation in hand, we note that if (h_1', h_2') dominates (h_1, h_2) then $\text{Max} \geqslant \text{Max}'$ and $\text{Min} \geqslant \text{Min}'$, which can be checked directly.

Limited Equity. P satisfies Limited Equity at (r_1, r_2) if, for all (h_1, h_2) and (h_1', h_2') such that there is no dominance relation between them, $g > g'$ implies $(h_1', h_2')P(h_1, h_2)$.

Equal-Sum Equity. P satisfies Equal-Sum Equity at (r_1, r_2) if, for all (h_1, h_2) and (h_1', h_2') such that $\Sigma = \Sigma'$, $g > g'$ implies $(h_1', h_2')P(h_1, h_2)$.

Observe that if (h_1', h_2') dominates (h_1, h_2) then $\Sigma > \Sigma'$. Thus, if the sums of two hospital pair rankings are equal ($\Sigma = \Sigma'$) then there is no dominance relation between them, and therefore Limited Equity implies Equal-Sum Equity. We will show next that the only pairing rule that satisfies Strong Pareto, Cross-Dominance and Limited Equity is a rule which closely resembles the leximin rule in contexts where agents are assumed to have interpersonally comparable utilities. Although we do not have utilities in our model, only ordinal rankings, if we want to make some interpersonal comparisons then the ordinal rank numbers have to be treated as utility levels that can be compared. We will define the Rank-Based Leximin rule in our setup next.

Rank-Based Leximin rule

Fix $(r_1, r_2) \in R \times R$ and let P denote the joint preference ordering $\varphi(r_1, r_2)$, where φ is the Rank-Based Leximin rule. Given (r_1, r_2), for (h_1, h_2) and $(h_1', h_2') \in H \times H$, let Max, Min, Max$'$ and Min$'$ be defined as above. Then $(h_1', h_2')P(h_1, h_2)$ if one of the following three cases holds:

1. $\text{Max} > \text{Max}'$;

2. $\text{Max} = \text{Max}'$ and $\text{Min} > \text{Min}'$;

3. $\text{Max} = \text{Max}'$, $\text{Min} = \text{Min}'$ and $r_1(h_1) > r_1(h_1')$.

In the definition we follow the convention that when comparing two symmetric pairs with ranks (x, y) and (y, x), where $x \neq y$, what we will refer to as *symmetric opposites* from now on, then the rule always favors doctor 1, that is, if $x < y$ then (x, y) is preferred to (y, x) in the aggregate preferences since doctor 1 prefers the hospital with rank x to the hospital with rank y. This implies that $\text{Max} = \text{Max}'$ and $\text{Min} = \text{Min}'$, and in this case $r_1(h_1) > r_1(h_1')$ leads to $(h_1', h_2')P(h_1, h_2)$. Although a less systematic favoring of one agent over the other would be slightly more equitable, we make this assumption for ease of exposition.

Our characterization of the Rank-Based Leximin rule is stated next. Note that since all three axioms are compatible with anonymity (the agents' names don't matter), the conjunction of these axioms does not necessarily break the "tie" between two hospital pairs that are symmetric opposites, and we use the same convention in the characterization as for the rule itself, favoring doctor 1 over doctor 2 in such cases. The proof of the theorem is in the Appendix.

Theorem 1. (Characterization of the Rank-Based Leximin rule)
A pairing rule satisfies Strong Pareto, Cross-Dominance and Limited Equity if and only if it is the Rank-Based Leximin rule.

Equivalently, we could also state that a pairing rules satisfies Dominance and Limited Equity if and only if it is the Rank-Based Leximin rule. The combination of the axioms in the theorem gives us a good intuitive idea about the Rank-Based Leximin rule. When there is a dominance relation between two hospital pairs, the rule ranks the dominating pair higher, indicating its efficiency properties. When there is no dominance relation between two hospital pairs, the rule ranks the hospital pair with the lower gap higher, which reflects that the aggregation rule prefers treating the two partners more equitably in these cases in terms of their preference ranks. Note also that Cross-Dominance has a rank-based fairness component as well, in the spirit of Rawls' 'behind a veil of ignorance' concept. It is also worth noting that the requirements of the axioms are compatible with each other in the sense that they lead to a transitive aggregate preference ordering.

Example 1. We provide examples of pairing rules to establish the independence of the three axioms in Theorem 1. The examples are given for $q = 4$ for simplicity, but similar examples can be found for higher numbers of hospitals as well. The orderings are in descending order of preference, and since they are indicated in terms of rank numbers, an ordering of all rank pairs entirely describes a pairing rule for all different rankings of hospitals.[2] The first column in Table 1 shows the Rank-Based Leximin rule, denoted by P^L, and the difference from this rule is indicated in bold in the other columns. $\bar{P}$ is an example where Limited Equity is not satisfied but Dominance is. This is an additive (or Borda) rule, since any pair with a lower sum is ranked ahead of a pair with a higher sum, which immediately implies that Dominance is satisfied. This rule also satisfies Equal-Sum Equity, which demonstrates that Limited Equity cannot be weakened to Equal-Sum Equity in the characterization in Theorem 1. $\tilde{P}$ is a pairing rule which satisfies Limited Equity and Cross-Dominance but not Strong Pareto, and $\hat{P}$ is a pairing rule which doesn't satisfy Cross-Dominance but satisfies the other two axioms. Let us note that although the differences from the Rank-Based Leximin

[2] This is possible because all these rules are neutral, that is, the hospitals' names don't matter.

rule may appear small in these examples, for larger examples with more hospitals more significant departures from the Rank-Based Leximin rule are possible.

$\boldsymbol{P^L}$	$\boldsymbol{\bar{P}}$	$\boldsymbol{\tilde{P}}$	$\boldsymbol{\hat{P}}$
1, 1	1, 1	1, 1	1, 1
1, 2	1, 2	1, 2	1, 2
2, 1	2, 1	2, 1	2, 1
2, 2	2, 2	2, 2	2, 2
1, 3	1, 3	**3, 1**	1, 3
3, 1	3, 1	**2, 3**	**2, 3**
2, 3	2, 3	**1, 3**	**3, 1**
3, 2	3, 2	3, 2	3, 2
3, 3	**1, 4**	3, 3	3, 3
1, 4	**4, 1**	1, 4	1, 4
4, 1	**3, 3**	4, 1	4, 1
2, 4	2, 4	**4, 2**	2, 4
4, 2	4, 2	**3, 4**	4, 2
3, 4	3, 4	**2, 4**	3, 4
4, 3	4, 3	4, 3	4, 3
4, 4	4, 4	4, 4	4, 4

Table 1: Independence of the axioms in Theorem 1

5 Quasi-Equitable Pairing Rules

We are also interested in rules that are less extreme than either the Lexicographic rule or the Rank-Based Leximin rule. Thus, we want to study a class of rules which includes both of these pairing rules, among others. These rules are necessarily lopsided, treating one agent better than the other, and we relax Limited Equity to allow for more asymmetry in the treatment of the two partners. We will follow the convention used before, which favors doctor 1 over doctor 2 when the rule treats the two partners asymmetrically.

Quasi-Equity. P satisfies Quasi-Equity at (r_1, r_2) if, for all (h_1, h_2) and (h'_1, h'_2) such that there is no dominance relation between them, $g > g'$ and $(h_1, h_2)P(h'_1, h'_2)$ imply that $r_1(h_1) < r_1(h'_1)$.

This axiom allows to escape the conclusion of Limited Equity, namely $(h'_1, h'_2)P(h_1, h_2)$, only if doctor 1 prefers h_1 to h'_1. It is easy to see that Limited Equity implies Quasi-Equity, and thus it follows from Theorem 1 that the Rank-Based Leximin rule satisfies Quasi-Equity. The Lexicographic rule (favoring doctor 1) also satisfies Quasi-Equity, which can be verified as follows. If $(h_1, h_2)P(h'_1, h'_2)$ for the Lexicographic rule at some (r_1, r_2) then $r_1(h_1) \leq r_1(h'_1)$ is implied immediately. To rule out $r_1(h_1) = r_1(h'_1)$, note that in this case there is a dominance relation between (h_1, h_2) and (h'_1, h'_2), contrary to the premise of the axiom. Therefore, since both the Rank-Based Leximin and the Lexicographic rules satisfy Strong Pareto, Strong Pareto together with Quasi-Equity captures a family of pairing rules which includes both of these rules. We call this family of pairing rules General Lexi-Pairing rules. We omit the straightforward proof which demonstrates that the General Lexi-Pairing rules, as defined below, are the only rules which satisfy Strong Pareto and Quasi-Equity.

General Lexi-Pairing rules

Fix $(r_1, r_2) \in R \times R$ and let P denote the paired preference ordering $\varphi(r_1, r_2)$, where φ is a General Lexi-Pairing rule.

Let (h_1, h_2), $(h'_1, h'_2) \in H \times H$. Then $(h'_1, h'_2)P(h_1, h_2)$ if either one of the following holds:

1. (h'_1, h'_2) Pareto-dominates (h_1, h_2);
2. there is no dominance relation between (h_1, h_2) and (h'_1, h'_2), $g > g'$, and $r_2(h_2) - r_1(h_1) < r_2(h'_2) - r_1(h'_1)$.

Proposition 2. *A pairing rule satisfies Strong Pareto and Quasi-Equity if and only if it is a General Lexi-Pairing rule.*

We can explain the family of General Lexi-Pairing rules intuitively in terms of a directed graph which shows all "immediate" Pareto-dominance relationships, where immediate means that only one doctor's allocation is different between the two hospital pairs, and this doctor's ranking of the two hospitals only differs by one, where the directed edge points toward the less preferred pair (with the higher rank number). This graph is presented in Figure 1 for the case of 4 hospitals, where the hospital pairs are represented by their rank numbers. If a hospital pair (h_1, h_2) can be reached from another hospital pair (h'_1, h'_2) by following a sequence of directed edges, then we call (h'_1, h'_2) a *predecessor* of (h_1, h_2) in the graph. Clearly, if (h'_1, h'_2) is a predecessor of (h_1, h_2) then it Pareto-dominates (h_1, h_2). Furthermore, we can see that the middle "column" (with $g = 0$) contains all pairs of the form (r, r), which are pairs that assign the same-ranked hospital to both partners, and hospital pairs to the left of this column are the hospital pairs which are better for doctor 1, while hospital pairs to the right of this middle column are better for doctor 2 in terms of the respective rank numbers.

The top-ranked pair for all General Lexi-Pairing rules is the pair of first-ranked hospitals for the two doctors respectively: $(1, 1)$. Point 1. in the definition of General Lexi-Pairing rules says that Strong Pareto is satisfied by the rule, and Strong Pareto implies that starting from $(1, 1)$ the pairing rule "traverses" the graph, although not necessarily following directed edges but picking up each vertex (i.e., each hospital pair), eventually ending this procedure with (q, q). This determines an ordering of the hospital pairs and thus identifies a neutral pairing rule. In order to ensure that Strong Pareto is satisfied, whenever the next vertex is picked each predecessor of this vertex in the graph has to have been picked already. Intuitively, the

Lexicographic rule "keeps to the left" in the graph when picking vertices, subject to Strong Pareto, given that the left side favors agent 1, while the Rank-Based Leximin rule "keeps to the middle."

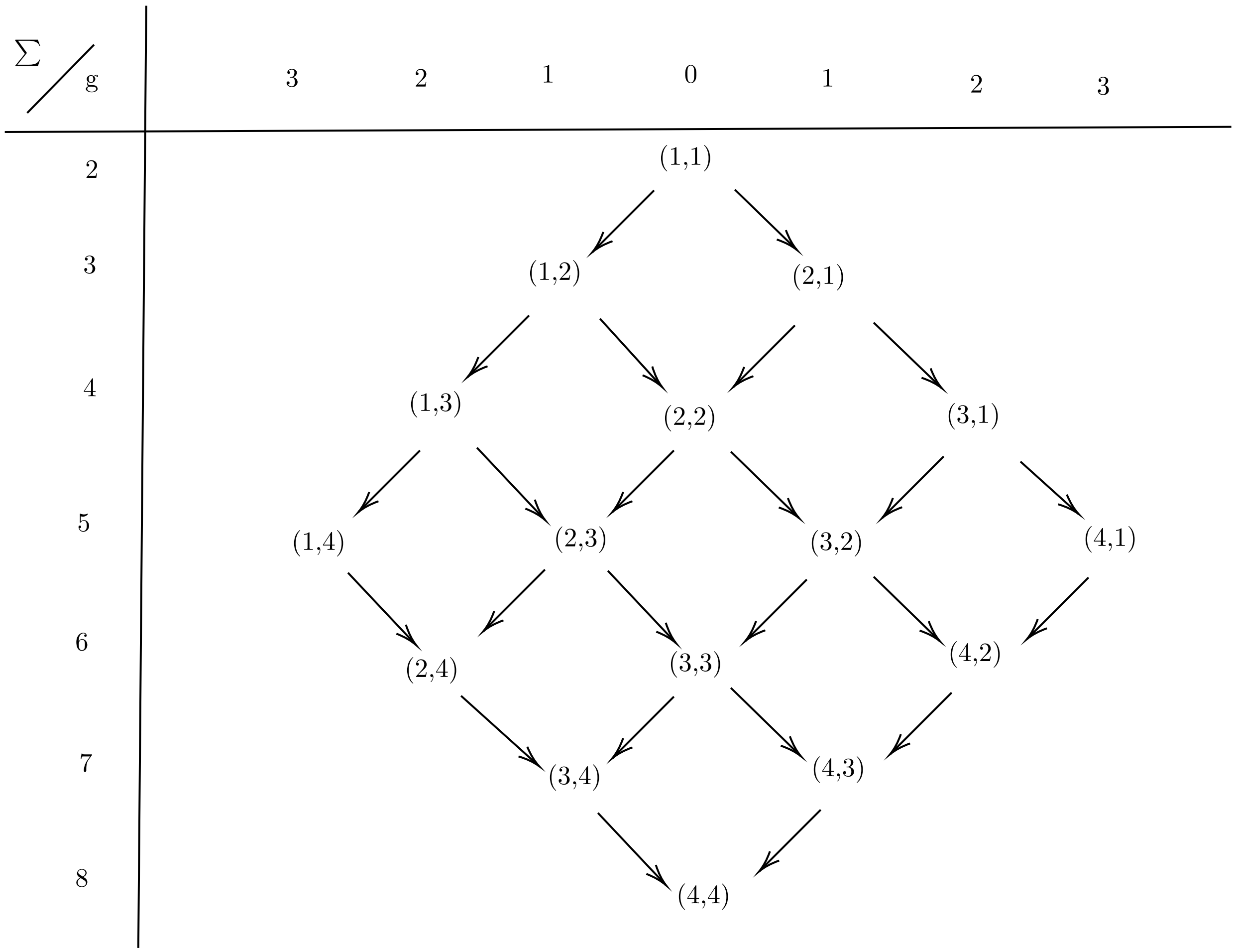

Figure 1: Immediate Pareto-dominance graph when $q = 4$

In order to find all General Lexi-Pairing rules, the only restriction regarding the order of picking vertices in the graph, in addition to Strong Pareto (i.e., ensuring that all predecessors in the graph had been already picked for each pair), as specified in point 2. of the definition of General Lexi-Pairing rules, is that if there is no dominance relation between (h_1, h_2) and (h'_1, h'_2), the gap is smaller for (h'_1, h'_2) than for (h_1, h_2), and (h'_1, h'_2) is to the left of (h_1, h_2) in the graph, indicating that agent 1 is favored relatively more by (h'_1, h'_2) than by (h_1, h_2), then (h'_1, h'_2) is ranked ahead of (h_1, h_2). Apart from this restriction, any other ordering of hospital pairs satisfying Strong Pareto leads to a General Lexi-Pairing rule. This restriction rules out, for example, the additive rule $\bar{P}$ presented in Example 1 since, for instance, this restriction implies that $(3, 3)$ is ranked ahead of $(4, 1)$ by any General Lexi-Pairing rule.

6 The Lexi-Pairing Rules

The family of General Lexi-Pairing rules is large and contains members that are not very desirable. One example of such a General Lexi-Pairing rule is the following. Let the aggregated preference ordering start with $(1,1,),(1,2),(1,3),(1,4)$, as in the Lexicographic rule, and let the rest of the ordering follow the ordering of the pairs in the Rank-Based Leximin rule. This pairing rule satisfies Strong Pareto (use the graph in Figure 1 to easily verify this) and Quasi-Equity. However, this rule is rather inconsistent in its treatment of the two partners. In order to gain some consistency for pairing rules and reduce this family of rules to a subfamily whose members possess a clear structure, we introduce two more (classes of) properties of pairing rules to ensure a consistent compromise between the partners. The extent of equity between the partners is represented by the parameter k.

***k*-Compromise.** Given $k \in \{0,\ldots,q-1\}$, P satisfies k-Compromise at $(r_1, r_2) \in R \times R$ if for all (h_1, h_2) and (h_1', h_2') such that $r_1(h_1) < r_1(h_1')$ and $r_2(h_2) > r_2(h_2')$:

1. $r_1(h_1') < r_2(h_2) - k$ implies that $(h_1', h_2')P(h_1, h_2)$, and
2. $r_1(h_1') > r_2(h_2) - k$ implies that $(h_1, h_2)P(h_1', h_2')$.

Unlike Strong Pareto, this axiom is relevant when there is a disagreement between the two doctors: doctor 1 prefers h_1 to h_1' and doctor 2 prefers h_2' to h_2. The parameter k shows the degree to which one partner is willing to compromise and take into account the other partner's preferences over hospital matches ahead of her own preferences and represents the degree of selfishness/altruism. In our exposition doctor 1 is always the selfish or favored partner and doctor 2 is always the altruistic or non-favored partner, but the symmetric case where doctor 1 is altruistic and doctor 2 is selfish applies equally.

The axiom focuses on the rank of the worse alternative for each doctor, h_1' and h_2 respectively in the definition. When $k = 0$, if h_1' has a lower rank number than h_2 then $(h_1', h_2')P(h_1, h_2)$ and vice versa, and there is as much compromise between the two doctors as possible. When $k = q - 1$ then doctor 1 is always favored and there is no compromise at all. In general, taking into account the selfishness/altrusim level k, if the less preferred alternative of doctor 1 has a relatively lower rank number compared to the rank number of the less preferred alternative of doctor 2 (in the two doctors' respective preferences), then the hospital pair less preferred by doctor 1 is ranked above the other hospital pair by the joint preference ordering P. In sum, this is an axiom that requires consistency regarding how each doctor compromises in favor of her partner, which is based on the ranking of alternatives. One can think of this as a consistent degree of selfishness or altruism, or "uniform" concession in terms of preference rank differences.

In order to show that k-Compromise is indeed a stronger requirement than Quasi-Equity, we prove the following result.

Proposition 3. *Let $k \in \{0, \dots, q-1\}$. Then k-Compromise implies Quasi-Equity.*

Proof. Let φ satisfy k-Compromise for some $k \in \{0, \dots, q-1\}$. Suppose that φ does not satisfy Quasi-Equity. Specifically, suppose that there exists $(r_1, r_2) \in R \times R$ such that for (h_1, h_2) and (h_1', h_2'), with no dominance relation between them, we have $g' > g$, $(h_1', h_2')P(h_1, h_2)$ where P denotes $\varphi(r_1, r_2)$, and $r_1(h_1') \geqslant r_1(h_1)$.

Since there is no Pareto-dominance, $r_1(h_1') > r_1(h_1)$, and thus no Pareto-dominance implies $r_2(h_2) > r_2(h_2')$. Then, if $r_1(h_1') > r_2(h_2) - k$, k-Compromise implies that $(h_1, h_2)P(h_1', h_2')$. Therefore, $r_1(h_1') \leqslant r_2(h_2) - k$, and hence no cross-dominance implies that $r_1(h_1') < r_2(h_2)$. In sum,

$$r_2(h_2) > r_1(h_1') > r_1(h_1). \tag{1}$$

Since $r_2(h_2) > r_1(h_1')$, no cross-dominance implies $r_2(h_2') > r_1(h_1)$. Then, in sum,

$$r_2(h_2) > r_2(h_2') > r_1(h_1). \tag{2}$$

Finally, note that (1) and (2) imply that $g' < g$, which is a contradiction. □

The next axiom pertains to the preference ordering of hospital pairs in the special case of a "tie," based on the selfishness/altruism parameter k.

***k*-Threshold-Consistency.** Given $k \in \{0, \dots, q-1\}$, P satisfies k-Threshold-Consistency at $(r_1, r_2) \in R \times R$ if for all (h_1, h_2) and (h_1', h_2') such that $r_1(h_1) < r_1(h_1')$, $r_2(h_2) > r_2(h_2')$ and $r_1(h_1') = r_2(h_2) - k$, $(h_1', h_2')P(h_1, h_2)$ if and only if $r_1(h_1) > r_2(h_2') - k$.

The axiom states that if the less-preferred hospital of doctor 1, h_1', relatively ties in terms of ranking with the less-preferred hospital of doctor 2's hospital, h_2, taking into account k as the degree of selfishness/altruism, then (h_1', h_2') is preferred to (h_1, h_2) if and only if the better option of doctor 1, h_1, has a higher rank number relatively, given k, than the better option of doctor 2, h_2'. For $k = 0$, when the less preferred alternative of each doctor has the same rank number, $(h_1', h_2')P(h_1, h_2)$ if and only if the more preferred alternative of doctor 1, h_1, has a rank number which is higher than the rank number of the more preferred alternative of doctor 2, h_2'.

These two axioms together with Strong Pareto lead to a parametric family of rules which includes both the Lexicographic and the Rank-Based Leximin rules. We call these pairing rules k-Lexi-Pairing rules (or Lexi-Pairing rules, for short).

k-Lexi-Pairing rules ψ^k ($k \in \{0, \ldots, q-1\}$)

Fix $k \in \{0, ..., q-1\}$ and $(r_1, r_2) \in R \times R$ and let P^k denote the paired preference ordering $\psi^k(r_1, r_2)$. For $(h_1, h_2) \in H \times H$, let

$$\text{Max} \equiv \max\,(r_1(h_1), r_2(h_2) - k);$$
$$\text{Min} \equiv \min\,(r_1(h_1), r_2(h_2) - k).$$

For $(h_1', h_2') \in H \times H$, let

$$\text{Max}' \equiv \max(r_1(h_1'), r_2(h_2') - k);$$
$$\text{Min}' \equiv \min\,(r_1(h_1'), r_2(h_2') - k).$$

Then $(h_1', h_2')P^k(h_1, h_2)$ if one of the following three cases holds:

1. $\text{Max} > \text{Max}'$;

2. $\text{Max} = \text{Max}'$ and $\text{Min} > \text{Min}'$;

3. $\text{Max} = \text{Max}'$, $\text{Min} = \text{Min}'$ and $r_1(h_1) > r_1(h_1')$.

It is important to note that the definition of k-Lexi-Pairing rules assigns a strict preference ordering P to every $(r_1, r_2) \in R \times R$ for each rule ψ^k, since the definition always specifies a strictly preferred hospital pair from any two distinct hospital pairs (h_1, h_2) and (h_1', h_2'): if $\text{Max} \neq \text{Max}'$ then case 1. applies, if $\text{Max} = \text{Max}'$ and $\text{Min} \neq \text{Min}'$ then case 2. applies, and if $\text{Max} = \text{Max}'$ and $\text{Min} = \text{Min}'$ then if both $r_1(h_1) = r_1(h_1')$ and $r_2(h_2) = r_2(h_2')$ then the two hospital pairs are the same, and if $r_1(h_1) = r_2(h_2')$, $r_2(h_2) = r_1(h_1')$, and $r_1(h_1) = r_1(h_1')$ then again the two hospital pairs are the same. Hence, we must have $r_1(h_1) = r_2(h_2')$, $r_2(h_2) = r_1(h_1')$, and $r_1(h_1) \neq r_1(h_1')$ (what we call symmetric opposites) and then case 3. applies. Moreover, transitivity of the preference relation can also be verified easily, since both the $>$ and the $\geqslant$ relations are transitive on the set of natural numbers.

There are q Lexi-Pairing rules in total, allowing a couple to choose from a range of q pairing rules which connect the seemingly unrelated Rank-Based Leximin and Lexicographic rules. When $k = 0$ the k-Lexi-Pairing rule is the Rank-Based Leximin rule, when $k = q - 1$ it is the Lexicographic rule, and all other Lexi-Pairing rules in-between are less extreme, where k represents the degree of selfishness of doctor 1 (and the degree of altruism of doctor 2). When comparing two hospital pairs in the two doctors' own respective preference rankings, k represents the willingness of doctor 1 to switch the order of the two hospital pairs in question when doctor 1 is relatively better off than doctor 2 in terms of individual hospital rankings in the preferred hospital pair, so that doctor 1 would be worse off and her partner would be better off after switching. As k increases, doctor 1 is less and less willing to make this switch, which makes doctor 1 more selfish and doctor 2 more altruistic. At the more symmetric end of the range, the Rank-Based Leximin rule is as fair between the two partners as possible within this family of pairing rules. There is no completely symmetric pairing rule in general, since in the case of symmetric opposites such as $(1,3)$ and $(3,1)$, one hospital pair has to be ranked above the other in a strict joint preference ordering. The Rank-Based Leximin pairing rule, as we defined it, always favors one partner over the other in such cases, but otherwise it treats the two doctors symmetrically in terms of their individual preference rankings and the asymmetric treatment between the two partners is minimal.

We present an example of Lexi-Pairing rules next. This example shows all four Lexi-Pairing rules in terms of the rank numbers when there are four hospitals.

Example 2. Let $q = 4$. Then there are four Lexi-Pairing rules when doctor 1 is selfish, corresponding to $k \in \{0, 1, 2, 3\}$. The top-ranked pair for all Lexi-Pairing rules (as noted before for General Lexi-Pairing rules) is $(1,1)$ and the last-ranked pair is $(4,4)$. The second-ranked pair is $(1,2)$ for each, but the third choice is $(2,1)$ when $k = 0$ (Rank-Based Leximin rule), and $(1,3)$ when $k = 3$ (Lexicographic rule). For each parameter k Table 2 displays the joint preference ordering P^k when doctor 1 is selfish and doctor 2 is altruistic.

$\boldsymbol{k=0}$	$\boldsymbol{k=1}$	$\boldsymbol{k=2}$	$\boldsymbol{k=3}$
$\boldsymbol{P^0}$	$\boldsymbol{P^1}$	$\boldsymbol{P^2}$	$\boldsymbol{P^3}$
1, 1	1, 1	1, 1	1, 1
1, 2	1, 2	1, 2	1, 2
2, 1	2, 1	1, 3	1, 3
2, 2	1, 3	2, 1	1, 4
1, 3	2, 2	2, 2	2, 1
3, 1	2, 3	1, 4	2, 2
2, 3	3, 1	2, 3	2, 3
3, 2	1, 4	2, 4	2, 4
3, 3	3, 2	3, 1	3, 1
1, 4	2, 4	3, 2	3, 2
4, 1	3, 3	3, 3	3, 3
2, 4	3, 4	3, 4	3, 4
4, 2	4, 1	4, 1	4, 1
3, 4	4, 2	4, 2	4, 2
4, 3	4, 3	4, 3	4, 3
4, 4	4, 4	4, 4	4, 4

Table 2: Lexi-Pairing rules when $q = 4$

The family of k-Lexi-Pairing rules consists of efficient and consistent pairing rules, each of which is uniquely described by the axioms of Strong Pareto, k-Compromise, and k-Threshold-Consistency. We will state our main result next, which provides a characterization of each of the Lexi-Pairing rules. The proof of the theorem is relegated to the Appendix.

Theorem 2. (Characterizations of Lexi-Pairing rules)
Let $k \in \{0, ..., q-1\}$. A pairing rule satisfies Strong Pareto, k-Compromise, and k-Threshold-Consistency if and only if it is the k-Lexi-Pairing rule.

Now we verify whether the three axioms in the theorem are independent of each other for $k \in \{0, \ldots, q-1\}$. Strong Pareto pertains to the pairwise ranking of hospital pairs over which the two doctors have no disagreement, while the other two axioms pertain to the pairwise ranking of hospital pairs over which the two doctors disagree. Hence, Strong Pareto is independent of the other two axioms, since comparisons when the partners are in agreement always need to be made. Similarly, k-Compromise is independent of the other two axioms, since comparisons when agents are not in agreement and their rank numbers do not tie always need to be made. However, it turns out that k-Threshold-Consistency is not needed when $k = q - 1$. k-Threshold-Consistency is used a lot when k is small, since the more symmetric treatment of the partners results in lots of ties, but as k increases there are fewer instances where such ties occur. For instance, we can see that in Example 1 there are 9 instances where the ranking of adjacent hospital pairs is determined by 0-Threshold-Consistency in the Rank-Based Leximin ordering P^0, there are 6 such instances in P^1 where 1-Threshold-Consistency is invoked, and only 2 instances in P^2 where 2-Threshold-Consistency is needed to pin down the joint preference ordering. Finally, the Lexicographic rule has zero such instances. This is always the case for the Lexicographic rule for an arbitrary number of hospitals q, because $r_1(h_1') = r_2(h_2) - k$ is never satisfied when $k = q - 1$, which can be seen as follows. Given that $r_1(h_1) < r_1(h_1')$, it follows that $r_1(h_1') \geqslant 2$, and since $r_2(h_2) \leqslant q$ we have $r_2(h_2) - (q - 1) \leqslant 1$. Thus, $r_1(h_1') \neq r_2(h_2) - (q - 1)$, and $(q-1)$-Threshold-Consistency is satisfied vacuously. It is clear, however, that $r_1(h_1') = r_2(h_2) - k$ is possible for all $k \in \{1, \ldots, q-2\}$, so k-Threshold-Consistency is only redundant for the Lexicographic rule and is needed for all the other Lexi-Pairing rules, as it is independent of the other two axioms for any k less than $q - 1$. Since preferences over hospital pairs are specified under each of these mutually exclusive scenarios by any well-defined pairing rule for all $k \in \{1, \ldots, q-2\}$, there is no redundant axiom in the corresponding characterizations.

Below we state two corollaries of Theorem 2 which show the characterizations of the Rank-Based Leximin and Lexicographic rules, providing alternative characterizations to Proposition 1 and Theorem 1. Let Compromise be the special case of k-Compromise with $k = 0$, and let Threshold-Consistency be the special case of k-Threshold-Consistency with $k = 0$. These axioms, which are used to characterize the Rank-Based Leximin rule, are particularly simple.

Corollary 1. *A pairing rule satisfies Strong Pareto, Compromise, and Threshold-Consistency if and only if it is the Rank-Based Leximin rule.*

Let Lexicographic-Compromise be the special case of k-Compromise with $k = q - 1$. Note that, as shown above, for $k = q - 1$ we have $r_1(h'_1) \geqslant 2$ and $r_2(h_2) - (q - 1) \leqslant 1$. Thus, $r_1(h'_1) > r_2(h_2) - k$ always holds and Lexicographic-Compromise (which offers no compromise at all, as we will see) can be stated simply as follows.

Lexicographic-Compromise. P satisfies Lexicographic-Compromise at $(r_1, r_2) \in R \times R$ if for all (h_1, h_2) and (h'_1, h'_2) such that $r_1(h_1) < r_1(h'_1)$ and $r_2(h_2) > r_2(h'_2)$, $(h_1, h_2)P(h'_1, h'_2)$.

This axiom together with Strong Pareto renders the characterization of the Lexicographic rule, stated below, straightforward.

Corollary 2. *A pairing rule satisfies Strong Pareto and Lexicographic-Compromise if and only if it is the Lexicographic rule.*

7 Geographic Constraints and Preferences for Being Together

A couple would typically prefer hospital positions that are close to each other, and in this section we explore the constraints that couples face when trying to find positions in the same geographic area. The geographic constraints may be given by a partition of the set of hospitals H, where each member represents a geographic area, or more generally we can use a graph, with the hospitals as vertices and the edges representing the compatibility of two hospitals in terms of togetherness for couples. It may be the case that hospitals h and h' are close enough to each other, and hospitals h' and h'' are also close enough to each other, to be acceptable to couples to have positions at h and h' respectively, and also at h' and h'' respectively, but not at h and h'', which would be deemed to be too far from each other. Also, some hospitals may be close enough to an airport with a good connection to some other airports, for example, and thus two hospitals near airports that are well connected would be considered compatible for a couple, but different hospitals further away from the airports in two different cities may not be considered compatible for a couple. We can summarize the hospital compatibility information by a set $G \subset H \times H$ which consists of compatible hospital pairs for couples (a set of ordered pairs of hospitals or, equivalently, a set of edges in the graph), and denote the set of incompatible hospital pairs by $\bar{G}$, where $G \cap \bar{G} = \emptyset$ and $G \cup \bar{G} = H \times H$. If $(h, h') \in G$ then getting positions at hospitals h and h' respectively for a couple is considered compatible in terms of geographic constraints, while if $(h, h'') \in \bar{G}$ then positions at h and h'' for a couple are not considered compatible. We assume that for all $(h, h') \in G$, $(h', h) \in G$ also holds. Also, for all $h \in H$, $(h, h) \in G$.

Although couples may have their own subjective opinions about which hospital pairs are close enough to be considered geographically compatible, for simplicity we take G to be a primitive of the model, and thus assume that any couple would deem the same hospital pairs compatible. Note that all previous papers that study a couple's preference for togetherness in the matching theory literature assume, at least implicitly, that geographical constraints are exogenously given, and our setup encompasses all different geographic considerations in the literature. One simple way is to assume that couples only find two positions close enough to each other if they are at the same hospital, as seen in Dutta and Massó (1997) and Khare and Roy (2018). This is a very specific case of our setup, where G is described by a partition of the set of hospitals into geographical areas with a single hospital in each geographical area. Another special case of our setup is explored by Cantala (2004) and Sethuraman et al. (2018), where hospitals are partitioned into regions and regions are assumed to have a common preference ranking by all couples.

We will introduce next two basic criteria that any paired preference ordering with geographic considerations based on G, denoted by P^G, should satisfy with respect to the paired preference ordering P which does not account for geographic constraints. These axioms together are related to the 'responsiveness violated for togetherness' (RVT) condition of Khare and Roy (2018), but in their paper togetherness always means that the couple gets placed at the same hospital.

Geographic Invariance. Let P and P^G be two paired preference orderings over $H \times H$. P^G satisfies Geographic Invariance with respect to P if the following two conditions hold:

1. for all $(h_1, h_2) \in G$ and $(h'_1, h'_2) \in G$, $(h_1, h_2)P^G(h'_1, h'_2)$ if and only if $(h_1, h_2)P(h'_1, h'_2)$;
2. for all $(h_1, h_2) \notin G$ and $(h'_1, h'_2) \notin G$, $(h_1, h_2)P^G(h'_1, h'_2)$ if and only if $(h_1, h_2)P(h'_1, h'_2)$.

Given two pairing rules φ and φ^G, φ^G satisfies Geographic Invariance with respect to φ if for all $(r_1, r_2) \in R \times R$, $\varphi^G(r_1, r_2)$ satisfies Geographic Invariance with respect to $\varphi(r_1, r_2)$.

Geographic Invariance expresses that the only valid preference reversals compared to the original paired preference ordering of the couple which does not take into account geographic preferences are ones based on the geographic constraints given by G.

Togetherness. Let P and P^G be two paired preference orderings over $H \times H$. P^G satisfies Togetherness with respect to P if for all $(h_1, h_2) \in G$ and $(h'_1, h'_2) \notin G$, if $(h_1, h_2)P(h'_1, h'_2)$ then $(h_1, h_2)P^G(h'_1, h'_2)$. Given two pairing rules φ and φ^G, φ^G satisfies Togetherness with respect to φ if for all $(r_1, r_2) \in R \times R$, $\varphi^G(r_1, r_2)$ satisfies Togetherness with respect to $\varphi(r_1, r_2)$.

The axiom of Togetherness requires that if a hospital pair is ranked by P ahead of another hospital pair, and the former pair is geographically compatible while the latter is not, then the former pair be still preferred by P^G to the latter, since the latter, due to the geographic incompatibility of the two hospitals, should only be ranked lower, not higher, when the joint preferences of the couple take into account geographic considerations.

Although G is not subject to a couple's subjective preferences and is assumed to be exogenously given, this doesn't mean that couples have to have the same kind of preferences over the geographic constraints. We allow couples to attribute different levels of importance to togetherness, and to this end introduce a togetherness parameter, denoted by t, which captures the extent to which a couple considers incompatible hospital pairs relatively less desirable when compared to compatible hospital pairs. Specifically, we use modified "rank numbers" based on t, denoted by $\hat{r}_i$ for doctor $i \in \{1, 2\}$, instead of the original rank numbers r_i. Although we still work with individual rank numbers of hospitals $\hat{r}_1$ and $\hat{r}_2$, due to the geographic constraints these are no longer the actual rank numbers of individual hospitals, instead, these are functions of a hospital pair, which allows for taking into account the geographic compatibility of the hospitals. Introducing the togetherness parameter t into k-Lexi-Pairing rules leads to the (k, t)-Couple-Lexi-Pairing rules as defined below. Note that in this more general framework the Lexi-Pairing preference orderings no longer satisfy responsiveness, as defined for couples by Khare et al. (2018), among others.

(k, t)-Couple-Lexi-Pairing rules $\psi^{(k,t)}$

Fix $k, t \in \{0, ..., q-1\}$ and let $(r_1, r_2) \in R \times R$.

Define $\hat{r}_1$ and $\hat{r}_2$ based on (r_1, r_2) as follows. Given $t \in \{0, \ldots, q-1\}$, for all $(h_1, h_2) \in H \times H$,

1. if $(h_1, h_2) \in G$ then $\hat{r}_1(h_1, h_2) = r_1(h_1)$ and $\hat{r}_2(h_1, h_2) = r_2(h_2)$;

2. if $(h_1, h_2) \in \bar{G}$ then $\hat{r}_1(h_1, h_2) = r_1(h_1) + t + \epsilon$ and $\hat{r}_2(h_1, h_2) = r_2(h_2) + t + \epsilon$, where $0 < \epsilon < 1$.

For $(h_1, h_2) \in H \times H$, let

$$\text{Max} \equiv \max\,(\hat{r}_1(h_1, h_2), \hat{r}_2(h_1, h_2) - k);$$
$$\text{Min} \equiv \min\,(\hat{r}_1(h_1, h_2), \hat{r}_2(h_1, h_2) - k).$$

For $(h'_1, h'_2) \in H \times H$, let

$$\text{Max}' \equiv \max(\hat{r}_1(h'_1, h'_2), \hat{r}_2(h'_1, h'_2) - k);$$
$$\text{Min}' \equiv \min\,(\hat{r}_1(h'_1, h'_2), \hat{r}_2(h'_1, h'_2) - k),$$

Let $P^{(k,t)}$ denote the paired preference ordering $\psi^{(k,t)}(r_1, r_2)$. Then $(h'_1, h'_2)P^{(k,t)}(h_1, h_2)$ if one of the following three cases holds:

1. Max $>$ Max$'$;

2. Max = Max′ and Min > Min′;

3. Max = Max′, Min = Min′, and $\hat{r}_1(h_1, h_2) > \hat{r}_1(h'_1, h'_2)$.

Geographically incompatible hospital pairs are less preferred, since their respective rank numbers are increased by $t + \epsilon$. Adding ϵ is needed to avoid potential ties in the rankings of hospital pairs, and the incompatible hospital pair is defined to be less preferred in case of a tie. For example, if the hospitals with original rank numbers $(1, 3)$ are incompatible (i.e., not in G) and $t = 2$, then without adding ϵ the rankings of these hospitals would become $(3, 5)$, which is the same as a compatible hospital pair (a hospital pair in G) that has the rank number pair $(3, 5)$ originally, and this would make these two hospital pairs indistinguishable when trying to rank them in the joint preferences. As a tie-breaker, adding ϵ causes the geographically compatible hospital pair to be preferred by both doctors to the geographically incompatible pair, but clearly this could be modified easily to reflect a preference for the incompatible hospital pair by subtracting ϵ instead of adding it.

The togetherness parameter $t \in \{0, \ldots, q-1\}$ expresses the preferences of a couple to obtain compatible positions in terms of geographic constraints. If $t = 0$ then the couple does not care about being together and the paired preference ordering is unchanged (since $\epsilon < 1$), regardless of G. At the other extreme, if $t = q - 1$ then each compatible pair of hospitals is preferred to each non-compatible pair of hospitals, while leaving all the other preference orderings unchanged. There are other cases between these two extremes which may more realistically depict a couple's preferences than either extremes, and the parameter t allows to systematically and consistently reduce the ranking of incompatible hospital pairs, while keeping other preference orderings the same. A higher value of t indicates that the couple finds it more important to find geographically compatible jobs, but note that the preference ordering may not necessarily change when the value of t changes, depending on the original preferences and on G.

It should be clear that $1 \leqslant \hat{r}_i \leqslant q + t + \epsilon$ and need not be natural numbers. While these are unusual "rank numbers," they allow us to simply apply the method of the Lexi-Pairing rules to these modified rank numbers which are based on G and the parameter t, expressing the couple's preference for togetherness.

Example 3. Let $H = \{a, b, c, d\}$. Since $q = 4$, there are four Lexi-Pairing rules when doctor 1 is selfish, as shown in Example 2. Let doctor 1's individual preference ordering over hospitals be $(a,\ b,\ c,\ d)$, where $r_1(a) = 1, r_1(b) = 2$, and so on. Let doctor 2's individual preference ordering over hospitals be $(a,\ c,\ d,\ b)$, where $r_2(a) = 1, r_2(c) = 2$, and so on. Assume also that hospitals a and b are in one geographic area, and hospitals c and d are in a different geographic area. Therefore, $G = \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d)\}$, where the first hospital is doctor 1's assigned hospital, and the second hospital is doctor 2's assigned hospital. Let the togetherness parameter be $t = 1$. Table 3 shows the k-Lexi-Pairing ordering for the given preferences favoring doctor 1 in the first column, denoted by P^k, and the $(k,1)$-Couple-Lexi-Pairing ordering for the given preferences favoring doctor 1 in the second column, denoted by $P^{(k,1)}$. The geographically incompatible hospital pairs are indicated in bold letters.

$\boldsymbol{k=0}$		$\boldsymbol{k=1}$		$\boldsymbol{k=2}$		$\boldsymbol{k=3}$	
$\boldsymbol{P^0}$	$\boldsymbol{P^{(0,1)}}$	$\boldsymbol{P^1}$	$\boldsymbol{P^{(1,1)}}$	$\boldsymbol{P^2}$	$\boldsymbol{P^{(2,1)}}$	$\boldsymbol{P^3}$	$\boldsymbol{P^{(3,1)}}$
a,a	a,a	a,a	a,a	a,a	a,a	a,a	a,a
$\boldsymbol{a,c}$	b,a	$\boldsymbol{a,c}$	b,a	$\boldsymbol{a,c}$	b,a	$\boldsymbol{a,c}$	a,b
b,a	$\boldsymbol{a,c}$	b,a	$\boldsymbol{a,c}$	$\boldsymbol{a,d}$	a,b	$\boldsymbol{a,d}$	b,a
$\boldsymbol{b,c}$	c,c	$\boldsymbol{a,d}$	a,b	b,a	$\boldsymbol{a,c}$	a,b	$\boldsymbol{a,c}$
$\boldsymbol{a,d}$	c,d	$\boldsymbol{b,c}$	c,c	$\boldsymbol{b,c}$	b,b	b,a	b,b
$\boldsymbol{c,a}$	$\boldsymbol{b,c}$	$\boldsymbol{b,d}$	b,b	a,b	$\boldsymbol{a,d}$	$\boldsymbol{b,c}$	$\boldsymbol{a,d}$
$\boldsymbol{b,d}$	a,b	$\boldsymbol{c,a}$	$\boldsymbol{a,d}$	$\boldsymbol{b,d}$	c,c	$\boldsymbol{b,d}$	c,c
c,c	b,b	a,b	c,d	b,b	c,d	b,b	c,d
c,d	$\boldsymbol{a,d}$	c,c	$\boldsymbol{b,c}$	$\boldsymbol{c,a}$	$\boldsymbol{b,c}$	$\boldsymbol{c,a}$	$\boldsymbol{b,c}$
a,b	d,c	b,b	$\boldsymbol{b,d}$	c,c	$\boldsymbol{b,d}$	c,c	$\boldsymbol{b,d}$
$\boldsymbol{d,a}$	$\boldsymbol{c,a}$	c,d	d,c	c,d	d,c	c,d	d,c
b,b	$\boldsymbol{b,d}$	$\boldsymbol{c,b}$	$\boldsymbol{c,a}$	$\boldsymbol{c,b}$	$\boldsymbol{c,a}$	$\boldsymbol{c,b}$	$\boldsymbol{c,a}$
d,c	d,d	$\boldsymbol{d,a}$	d,d	$\boldsymbol{d,a}$	d,d	$\boldsymbol{d,a}$	d,d
$\boldsymbol{c,b}$	$\boldsymbol{d,a}$	d,c	$\boldsymbol{c,b}$	d,c	$\boldsymbol{c,b}$	d,c	$\boldsymbol{c,b}$
d,d	$\boldsymbol{c,b}$	d,d	$\boldsymbol{d,a}$	d,d	$\boldsymbol{d,a}$	d,d	$\boldsymbol{d,a}$
$\boldsymbol{d,b}$	$\boldsymbol{d,b}$	$\boldsymbol{d,b}$	$\boldsymbol{d,b}$	$\boldsymbol{d,b}$	$\boldsymbol{d,b}$	$\boldsymbol{d,b}$	$\boldsymbol{d,b}$

Table 3: Couple-Lexi-Pairing orderings when $q=4$ and $t=1$

Now we modify the axioms that characterize the Lexi-Pairing rules, so that they reflect the preferences of couples for being together, given the geographic constraints represented by G. For each of the three axioms below and for each $(r_1, r_2) \in R \times R$, define $\hat{r}_1^t$ and $\hat{r}_2^t$ as a function of t, as before. Thus, for all $t \in \{0, \dots, q-1\}$:

1. if $(h_1, h_2) \in G$ then $\hat{r}_1^t(h_1, h_2) = r_1(h_1)$ and $\hat{r}_2^t(h_1, h_2) = r_2(h_2)$;

2. if $(h_1, h_2) \in \bar{G}$ then $\hat{r}_1^t(h_1, h_2) = r_1(h_1) + t + \epsilon$ and $\hat{r}_2^t(h_1, h_2) = r_2(h_2) + t + \epsilon$, where $0 < \epsilon < 1$.

***t*-Strong Pareto** (Strong Pareto with t-togetherness). Given $t \in \{0, \dots, q-1\}$, P satisfies t-Strong Pareto at $(r_1, r_2) \in R \times R$ if, for all (h_1, h_2) and (h_1', h_2') such that $\hat{r}_1^t(h_1, h_2) \geqslant \hat{r}_1^t(h_1', h_2')$ and $\hat{r}_2^t(h_1, h_2) \geqslant \hat{r}_2^t(h_1', h_2')$ with at least one strict inequality, $(h_1', h_2') P (h_1, h_2)$.

(k,t)-Compromise (k-Compromise with t-togetherness). Given $k, t \in \{0, \dots, q-1\}$, P satisfies (k,t)-Compromise at $(r_1, r_2) \in R \times R$ if for all (h_1, h_2) and (h_1', h_2') such that $\hat{r}_1^t(h_1, h_2) < \hat{r}_1^t(h_1', h_2')$ and $\hat{r}_2^t(h_1, h_2) > \hat{r}_2^t(h_1', h_2')$:

1. $\hat{r}_2^t(h_1', h_2') < \hat{r}_1^t(h_1, h_2) - k$ implies that $(h_1', h_2') P (h_1, h_2)$, and

2. $\hat{r}_2^t(h_1', h_2') > \hat{r}_1^t(h_1, h_2) - k$ implies that $(h_1, h_2)P(h_1', h_2')$.

(k,t)-Threshold-Consistency (k-Threshold-Consistency with t-togetherness). Given $k, t \in \{0, \ldots, q-1\}$, P satisfies (k,t)-Threshold-Consistency at $(r_1, r_2) \in R \times R$ if for all (h_1, h_2) and (h_1', h_2') such that $\hat{r}_1^t(h_1, h_2) < \hat{r}_1^t(h_1', h_2')$, $\hat{r}_2^t(h_1, h_2) > \hat{r}_2^t(h_1', h_2')$ and $\hat{r}_1^t(h_1', h_2') = \hat{r}_2^t(h_1, h_2) - k$, $(h_1', h_2')P(h_1, h_2)$ if and only if $\hat{r}_1^t(h_1, h_2) > \hat{r}_2^t(h_1', h_2') - k$.

Each of the above axioms, given a fixed togetherness parameter $t \in \{1, \ldots, q-1\}$ and G, is defined for a specific pair of individual rankings (r_1, r_2), and we will say that a pairing rule φ satisfies an axiom if for all $(r_1, r_2) \in R \times R$, the paired ranking $\varphi(r_1, r_2)$ satisfies the axiom at (r_1, r_2).

Proposition 4. (Properties of Couple-Lexi-Pairing rules)
Let $k \in \{0, \ldots, q-1\}$.

1. *For all $t \in \{0, \ldots, q-1\}$, a pairing rule satisfies t-Strong Pareto, (k,t)-Compromise, and (k,t)-Threshold-Consistency if and only if it is the (k,t)-Couple-Lexi-Pairing rule.*

2. *For all $t, \bar{t} \in \{0, \ldots, q-1\}$, the (k,t)-Couple-Lexi-Pairing rule satisfies Geographic Invariance with respect to the $(k,\bar{t})$-Couple-Lexi-Pairing rule.*

3. *For all $t, \bar{t} \in \{0, \ldots, q-1\}$ such that $t < \bar{t}$, the $(k,\bar{t})$-Couple-Lexi-Pairing rule satisfies Togetherness with respect to the (k,t)-Couple-Lexi-Pairing rule.*

Proof.

1. *Characterization.* This is a straightforward extension of the characterization in Theorem 2, since it can be seen easily that the proof of Theorem 2 holds for any pair of rank numbers associated with paired hospital positions, and need not be the rank numbers of individual hospital positions, as in Theorem 2. Although the modified $\hat{r}_i(h_1, h_2)$ rank numbers for $i \in \{1, 2\}$ may not be natural numbers, the proof of Theorem 2 still holds with the modified rank numbers as long as no two distinct hospital pairs have identical rank numbers. This condition is automatically satisfied when the rank numbers are simply the individual hospital ranks in the two respective individual preference orderings, but when more general rank numbers are allowed such a tie may occur, which would make it impossible to distinguish between the two hospital pairs with the same rank numbers for both hospital positions. However, given that such ties cannot exist if the original individual rank numbers $r_i(h_i)$ of hospitals are used, a tie could only occur between a hospital pair $(h_1, h_2) \in G$ and $(h_1', h_2') \notin G$, and only when a given fixed $t \in \{0, \ldots, q-1\}$ is added to the ranking of the incompatible hospitals h_1' and h_2'. Therefore, ties

are prevented by the addition of ϵ to the rank numbers of h'_1 and h'_2 and we can apply the proof of Theorem 2 to obtain this characterization result.

2. *Geographic Invariance.* Fix $(r_1, r_2) \in R \times R$ and let $P^{(k,t)}$ denote $\psi^{(k,t)}(r_1, r_2)$, where $\psi^{(k,t)}$ is the (k,t)-Couple-Lexi-Pairing rule, and let $P^{(k,\bar{t})}$ denote $\psi^{(k,\bar{t})}(r_1, r_2)$, where $\psi^{(k,\bar{t})}$ is the $(k,\bar{t})$-Couple-Lexi-Pairing rule. If $(h_1, h_2) \in G$ and $(h'_1, h'_2) \in G$ then $\hat{r}^t_1(h_1, h_2) = r_1(h_1)$, $\hat{r}^t_2(h_1, h_2) = r_2(h_2)$, $\hat{r}^t_1(h'_1, h'_2) = r_1(h'_1)$, $\hat{r}^t_2(h'_1, h'_2) = r_2(h'_2)$, $\hat{r}^{\bar{t}}_1(h_1, h_2) = r_1(h_1)$, $\hat{r}^{\bar{t}}_2(h_1, h_2) = r_2(h_2)$, $\hat{r}^{\bar{t}}_1(h'_1, h'_2) = r_1(h'_1)$, and $\hat{r}^{\bar{t}}_2(h'_1, h'_2) = r_2(h'_2)$. Thus, $(h_1, h_2)P^{(k,\bar{t})}(h'_1, h'_2)$ if and only if $(h_1, h_2)P^{(k,t)}(h'_1, h'_2)$. If $(h_1, h_2) \notin G$ and $(h'_1, h'_2) \notin G$ then $\hat{r}^t_1(h_1, h_2) = r_1(h_1) + t + \epsilon$, $\hat{r}^t_2(h_1, h_2) = r_2(h_2) + t + \epsilon$, $\hat{r}^t_1(h'_1, h'_2) = r_1(h'_1) + t + \epsilon$, $\hat{r}^t_2(h'_1, h'_2) = r_2(h'_2) + t + \epsilon$, $\hat{r}^{\bar{t}}_1(h_1, h_2) = r_1(h_1) + \bar{t} + \epsilon$, $\hat{r}^{\bar{t}}_2(h_1, h_2) = r_2(h_2) + \bar{t} + \epsilon$, $\hat{r}^{\bar{t}}_1(h'_1, h'_2) = r_1(h'_1) + \bar{t} + \epsilon$, and $\hat{r}^{\bar{t}}_2(h'_1, h'_2) = r_2(h'_2) + \bar{t} + \epsilon$, where $0 < \epsilon < 1$. Since adding a constant preserves the Max, the Min and the rank comparisons, $(h_1, h_2)P^{(k,\bar{t})}(h'_1, h'_2)$ if and only if $(h_1, h_2)P^{(k,t)}(h'_1, h'_2)$.

3. *Togetherness.* Fix $(r_1, r_2) \in R \times R$ and let $P^{(k,t)}$ denote $\psi^{(k,t)}(r_1, r_2)$, where $\psi^{(k,t)}$ is the (k,t)-Couple-Lexi-Pairing rule, and let let $P^{(k,\bar{t})}$ denote $\psi^{(k,\bar{t})}(r_1, r_2)$, where $\psi^{(k,\bar{t})}$ is the $(k,\bar{t})$-Couple-Lexi-Pairing rule and $t < \bar{t}$. If $(h_1, h_2) \in G$ and $(h'_1, h'_2) \notin G$ then $\hat{r}^t_1(h_1, h_2) = r_1(h_1)$, $\hat{r}^t_2(h_1, h_2) = r_2(h_2)$, $\hat{r}^t_1(h'_1, h'_2) = r_1(h'_1) + t + \epsilon$ and $\hat{r}^t_2(h'_1, h'_2) = r_2(h'_2) + t + \epsilon$, $\hat{r}^{\bar{t}}_1(h_1, h_2) = r_1(h_1)$, $\hat{r}^{\bar{t}}_2(h_1, h_2) = r_2(h_2)$, $\hat{r}^{\bar{t}}_1(h'_1, h'_2) = r_1(h'_1) + \bar{t} + \epsilon$ and $\hat{r}^{\bar{t}}_2(h'_1, h'_2) = r_2(h'_2) + \bar{t} + \epsilon$, where $0 < \epsilon < 1$. Thus, given the definition of (k,t)-Couple-Lexi-Pairing rules, it is straightforward to check that if $(h_1, h_2)P^{(k,t)}(h'_1, h'_2)$ then $(h_1, h_2)P^{(k,\bar{t})}(h'_1, h'_2)$. □

8 Concluding Remarks

Apart from (k,t)-Couple-Lexi-Pairing rules, there are other natural ways to define preference aggregation rules for couples who take into account geographic constraints. We propose the Lexi-Pairing rules which have appealing efficiency and fairness properties and are very consistent in terms of the compromises made between the two partners. Geographic preferences may also be defined differently. For example, we could let couples leave top-ranked hospital pairs where they originally are in the joint preference ordering and only reduce the joint ranking of individually lower-ranked hospital pairs, expressing that a couple is less willing to sacrifice their individual hospital choices if they can get very highly preferred hospitals, even when these hospitals are not in the same geographic area. Our setup could also allow the two partners to use two different t-parameters which are added to the rank numbers of incompatible hospital pairs, expressing that one partner may value togetherness more than the other (although this could lead to divorce). Thus, if $t_1 > t_2$ then doctor 1 values togetherness more than doctor 2, and vice versa.

While there are many ways to aggregate couples' individual preferences over pairs of jobs, our intention was to propose and study specific intuitively appealing aggregation methods. According to our proposed family of rules, if a couple is not sure about how to rank the job pairs but know their individual rankings over jobs, the two partners would only need to negotiate about the compromise parameter k and agree on their preferences over the level of togetherness to determine parameter t. While this imposes constraints on the couple's joint preference choices, the proposed Couple-Lexi-Pairing rules offer a simple way to generate systematically aggregated paired preference rankings, and the preference aggregation choices are clarified by their properties that are shown in this paper.

It is also important to note that the proposed (k, t)-Couple-Lexi-Pairing rules are informationally simple when considering a couple's reported preferences as an input to a centralized matching system. When submitting the preferences of the couple to a clearing house such as the NRMP, a couple would not need to report an entire preference ordering over hospital pairs, which may be cumbersome and could lead to listing fewer hospital pairs than acceptable to the couple. Rather, they would report their individual rankings over hospitals, just like any single applicant, and in addition they would only need to report two parameters: k, to specify their joint compromise over individual rankings, and t, to indicate their joint preference for geographic proximity. Furthermore, the design of the matching mechanism may be able to exploit the simple structure and clarity of couples' preferences and produce more desirable outcomes for matching markets with couples.

Appendix

Proof of Theorem 1

Claim 1.1. *The Rank-Based Leximin rule satisfies Limited Equity.*

Proof. Fix $(r_1, r_2) \in R \times R$ and let P denote $\varphi(r_1, r_2)$, where φ is the Rank-Based Leximin Rule. Let (h_1, h_2) and (h'_1, h'_2) be such that there is no dominance relation between them and $g > g'$. We will show that then $(h'_1, h'_2)P(h_1, h_2)$.

Case 1: $\text{Max} > \text{Max}'$.

Then $(h'_1, h'_2)P(h_1, h_2)$.

Case 2: $\text{Max} < \text{Max}'$

Since $g > g'$, we must have $\text{Min} < \text{Min}'$. This implies that (h_1, h_2) dominates (h'_1, h'_2), which is a contradiction.

Case 3: $\text{Max} = \text{Max}'$

Then if $\text{Min}' \neq \text{Min}$ there is a dominance relation between (h_1, h_2) and (h'_1, h'_2), and thus $\text{Min} = \text{Min}'$. This implies that $g = g'$, which is a contradiction. □

Claim 1.2. *The Rank-Based Leximin rule satisfies Cross-Dominance.*

Proof. Fix $(r_1, r_2) \in R \times R$ and let P denote $\varphi(r_1, r_2)$, where φ is the Rank-Based Leximin Rule. Let (h_1, h_2) and (h'_1, h'_2) be such that $r_1(h_1) \geqslant r_2(h'_2)$ and $r_2(h_2) \geqslant r_1(h'_1)$ with at least on inequality. We will show that then $(h'_1, h'_2)P(h_1, h_2)$.

Case 1: $\text{Max} > \text{Max}'$

Then $(h'_1, h'_2)P(h_1, h_2)$.

Case 2: $\text{Max} < \text{Max}'$

Subcase 2.1: If $\text{Max}' = r_1(h'_1)$ then, since $r_2(h_2) \geqslant r_1(h'_1)$, $Max = r_1(h_1)$ and $r_2(h_2) \geqslant r_1(h'_1) > r_1(h_1)$. Thus, $\text{Max} = r_2(h_2)$, which is a contradiction.

Subcase 2.2: If $\text{Max}' = r_2(h'_2)$ then, since $r_1(h_1) \geqslant r_2(h'_2)$, $\text{Max} = r_2(h_2)$ and $r_1(h_1) \geqslant r_2(h'_2) > r_2(h_2)$. Thus, $\text{Max} = r_1(h_1)$, which is a contradiction.

Case 3: $\text{Max} = \text{Max}'$

If $\text{Min} > \text{Min}'$ then $(h'_1, h'_2)P(h_1, h_2)$. Assume that $\text{Min} \leqslant \text{Min}'$.

Subcase 3.1: If $\text{Min} = r_1(h_1)$ then either a) $r_1(h_1) = r_2(h'_2)$ and $\text{Min}' = r_2(h'_2)$ or b) $\text{Min}' = r_1(h'_1)$. If a) holds then $r_2(h_2) > r_1(h'_1)$ and then $\text{Max}' \geqslant \text{Min}'$ implies $r_2(h_2) > r_1(h'_1) \geqslant r_2(h'_2) = r_1(h_1)$. Then Max $= r_2(h_2) \neq \text{Max}'$, which is a contradiction. Thus, b) holds. Therefore, $\text{Min}' = r_1(h'_1)$ and $r_1(h'_1) \geqslant r_1(h_1) \geqslant r_2(h'_2)$. Thus, $\text{Min}' = r_2(h'_2)$. Since $\text{Min}' = r_1(h'_1)$, $r_2(h_2) \geqslant r_1(h'_1) = r_1(h_1) = r_2(h'_2)$. Then $r_2(h_2) > r_1(h'_1)$. However, this contradicts $\text{Max} = \text{Max}'$.

Subcase 3.2: If $\text{Min} = r_2(h_2)$ then either a) $r_2(h_2) = r_1(h'_1)$ and $\text{Min}' = r_1(h'_1)$ or b) $\text{Min}' = r_2(h'_2)$. If a) holds then $r_1(h_1) > r_2(h'_2)$ and then $\text{Max}' \geqslant \text{Min}'$ implies $r_1(h_1) > r_2(h'_2) \geqslant r_1(h'_1) = r_2(h_2)$. Then Max $= r_1(h_1) \neq \text{Max}'$, which is a contradiction. Thus, b) holds. Therefore, $\text{Min}' = r_2(h'_2)$ and $r_2(h'_2) \geqslant r_2(h_2) \geqslant r_1(h'_1)$. Thus, $\text{Min}' = r_1(h'_1)$. Since $\text{Min}' = r_2(h'_2)$, $r_1(h_1) \geqslant r_2(h'_2) = r_2(h_2) = r_1(h'_1)$. Then $r_1(h_1) > r_2(h'_2)$. However, this contradicts $\text{Max} = \text{Max}'$. □

Claim 1.3. *If a pairing rule satisfies Strong Pareto, Cross-Dominance and Limited Equity, then it is the Rank-Based Leximin rule.*

Proof. Fix $(r_1, r_2) \in R \times R$ and let P denote $\varphi(r_1, r_2)$, where φ satisfies Strong Pareto, Cross-Dominance and Limited Equity. Let (h_1, h_2) and (h'_1, h'_2) be two distinct hospital pairs.

If $\text{Max} = \text{Max}'$ and $\text{Min} > \text{Min}'$, (h'_1, h'_2) dominates (h_1, h_2). Then Dominance implies that $(h'_1, h'_2)P(h_1, h_2)$.

If $\text{Max} = \text{Max}'$ and $\text{Min} = \text{Min}'$ then, given that $(h_1, h_2) \neq (h'_1, h'_2)$, we have symmetric opposites: $r_1(h_1) = r_2(h'_2)$, $r_2(h_2) = r_1(h'_1)$ and $r_1(h_1) \neq r_1(h'_1)$. Then, if $r_1(h_1) > r_1(h'_1)$ then $(h'_1, h'_2)P(h_1, h_2)$ by convention.

If Max $>$ Max′, we consider three cases regarding $\sum$ and $\sum'$. Note first that Max $=$ Min $+g = \sum -$ Min, and hence 2Max $= \sum + g$. Similarly, 2Max′ $= \sum' + g'$. Thus Max $>$ Max′ if and only if $\sum + g > \sum' + g'$.

Case 1: $\sum = \sum'$

Then $\sum + g > \sum' + g'$ implies $g > g'$, and thus $(h'_1, h'_2)P(h_1, h_2)$ by Equal-Sum Equity.

Case 2: $\sum > \sum'$

Let $(\tilde{h}_1, \tilde{h}_2)$ be such that $\tilde{\text{Max}} = \text{Max}' + 1$ and $\tilde{\text{Min}} = \sum - \text{Max}' - 1$. Note that $\tilde{\text{Min}}$ is feasible, since $\text{Min}' \geqslant 1$, $\sum' \geqslant \text{Max}' + 1$ and thus $\sum > \sum'$ implies that $\sum > \text{Max}' + 1$, and hence $\tilde{\text{Min}} > 0$. Observe that $\tilde{\sum} = \sum$. We will show next that $\tilde{g} \leqslant g$. Given that $\text{Max}' < \text{Max}$, $\text{Max}' + 1 \leqslant \text{Max}$. Then $2\text{Max}' + 2 \leqslant 2\text{Max} = \text{Max} - \text{Min} + \text{Max} + \text{Min}$. This means that $2\text{Max}' + 2 - \sum \leqslant \text{Max} - \text{Min}$, which is equivalent to $\tilde{g} \leqslant g$, since $\tilde{g} = \tilde{\text{Max}} - \tilde{\text{Min}} = \text{Max}' + 1 - \sum + \text{Max}' + 1 = 2\text{Max}' + 2 - \sum$ and $g = \text{Max} - \text{Min}$.

Now note that $\text{Max}' < \tilde{\text{Max}}$. We will show that $\text{Min}' \leqslant \tilde{\text{Min}}$. Since $\sum > \sum'$, $\sum' + 1 \leqslant \sum$. Thus, $\text{Min}' + \text{Max}' + 1 \leqslant \sum$ and $\text{Min}' \leqslant \sum - \text{Max}' - 1 = \tilde{\text{Min}}$. Therefore, (h'_1, h'_2) dominates $(\tilde{h}_1, \tilde{h}_2)$ and Dominance implies that $(h'_1, h'_2)P(\tilde{h}_1, \tilde{h}_2)$.

If $\tilde{g} < g$ then, given that $\tilde{\sum} = \sum$, Equal-Sum Equity implies $(\tilde{h}_1, \tilde{h}_2)P(h_1, h_2)$. If $\tilde{g} = g$, either $(\tilde{h}_1, \tilde{h}_2) = (h_1, h_2)$ or $(\tilde{h}_1, \tilde{h}_2)$ and (h_1, h_2) are symmetric opposites. Since (h'_1, h'_2) dominates $(\tilde{h}_1, \tilde{h}_2)$, it follows that in both cases (h'_1, h'_2) dominates (h_1, h_2), and therefore $(h'_1, h'_2)P(h_1, h_2)$ by Dominance.

Case 3: $\sum < \sum'$

Given that $\sum < \sum'$, (h'_1, h'_2) does not dominate (h_1, h_2). Since Max $>$ Max′, (h_1, h_2) does not dominate (h'_1, h'_2). Now note that $\sum + g > \sum' + g'$ implies that $g > g'$. Thus, given that there is no dominance relation between (h_1, h_2) and (h'_1, h'_2), Limited Equity implies that $(h'_1, h'_2)P(h_1, h_2)$. □

Finally, note that it follows from Claim 2.1 below that the Rank-Based Leximin rule satisfies Strong Pareto (the $k = 0$ case in Claim 2.1). Together with this result, Claims 1.1, 1.2, and 1.3 prove Theorem 1.

Proof of Theorem 2

Claim 2.1. *For all $k \in \{0, ..., q-1\}$ the k-Lexi-Pairing rule ψ^k satisfies Strong Pareto.*

Proof. Let $k \in \{0, ..., q-1\}$. Fix $(r_1, r_2) \in R \times R$ and let P^k denote $\psi^k(r_1, r_2)$. Let (h_1, h_2) and (h'_1, h'_2) satisfy $r_1(h_1) \geqslant r_1(h'_1)$ and $r_2(h_2) \geqslant r_2(h'_2)$, with at least one strict inequality. We need to show that $(h'_1, h'_2)P^k(h_1, h_2)$.

Case 1. $r_1(h_1) > r_1(h'_1)$ and $r_2(h_2) > r_2(h'_2)$

Subcase 1.1: If $r_1(h_1) \geqslant r_2(h_2) - k$ then $\text{Max} = r_1(h_1) > r_1(h'_1)$, and $\text{Max} > r_2(h'_2) - k$. Thus, Max $>$ Max′.

Subcase 1.2: If $r_1(h_1) < r_2(h_2) - k$ then Max $= r_2(h_2) - k > r_1(h_1) > r_1(h_1')$, and Max$> r_2(h_2') - k$. Thus, Max $>$ Max$'$.

Case 2. $r_1(h_1) = r_1(h_1')$ and $r_2(h_2) > r_2(h_2')$

Subcase 2.1: If $r_1(h_1) \geqslant r_2(h_2) - k$, then Max $= r_1(h_1) = r_1(h_1')$ and Max $> r_2(h_2') - k$. Thus, Max$' = r_1(h_1')$ and Max $=$ Max$'$. Furthermore, Min $= r_2(h_2) - k$, Min $> r_2(h_2') - k$, and since Max$' = r_1(h_1')$, Min$' = r_2(h_2') - k$. Thus, Min $>$ Min$'$.

Subcase 2.2: If $r_1(h_1) < r_2(h_2) - k$ then Max $= r_2(h_2) - k$, Max $> r_2(h_2') - k > r_1(h_1) = r_1(h_1')$. Thus, Max $>$ Max$'$.

Case 3. $r_1(h_1) > r_1(h_1')$ and $r_2(h_2) = r_2(h_2')$

Subcase 3.1: If $r_1(h_1) > r_2(h_2) - k$ then Max $= r_1(h_1) > r_1(h_1')$, and Max $> r_2(h_2') - k$. Thus, Max $>$ Max$'$.

Subcase 3.2: If $r_1(h_1) \leqslant r_2(h_2) - k$ then Max $= r_2(h_2) - k = r_2(h_2') - k \geqslant r_1(h_1) > r_1(h_1')$. Thus, Max $=$ Max$'$. Then, Min $= r_1(h_1)$ and Min $> r_1(h_1')$, which implies that Min $>$ Min$'$.

Therefore, $(h_1', h_2')P^k(h_1, h_2)$ in each case. Since this holds for all $(r_1, r_2) \in R \times R$, for all $k \in \{0, \ldots, q-1\}$, the k-Lexi-Pairing rule satisfies Strong Pareto. □

Claim 2.2. *For all $k \in \{0, \ldots, q-1\}$, the k-Lexi-Pairing rule satisfies k-Compromise.*

Proof. Let $k \in \{0, \ldots, q-1\}$. Fix $(r_1, r_2) \in R \times R$ and let P^k denote $\psi^k(r_1, r_2)$. Let (h_1, h_2) and (h_1', h_2') satisfy $r_1(h_1) < r_1(h_1')$ and $r_2(h_2) > r_2(h_2')$.

Case 1: We will show that $r_1(h_1') < r_2(h_2) - k$ implies $(h_1', h_2')P^k(h_1, h_2)$.

Note first that $r_2(h_2) - k > r_1(h_1') > r_1(h_1)$ and thus Max $= r_2(h_2) - k$.

Subcase 2.1: If Max$' = r_1(h_1')$ then $r_2(h_2) - k > r_1(h_1')$ implies Max $>$ Max$'$.

Subcase 2.2: If Max$' = r_2(h_2') - k$ then $r_2(h_2) > r_2(h_2')$ implies Max $>$ Max$'$.

Hence, $(h_1', h_2')P^k(h_1, h_2)$ in both subcases.

Case 2: We will show that $r_1(h_1') > r_2(h_2) - k$ implies $(h_1, h_2)P^k(h_1', h_2')$.

Note first that $r_1(h_1') + k > r_2(h_2) > r_2(h_2')$ and thus Max$' = r_1(h_1')$.

Subcase 1.1: If Max $= r_1(h_1)$ then $r_1(h_1) < r_1(h_1')$ implies Max $<$ Max$'$.

Subcase 1.2: If Max $= r_2(h_2) - k$ then $r_2(h_2) - k < r_1(h_1')$ implies Max $<$ Max$'$.

Hence, $(h_1, h_2)P^k(h_1', h_2')$ in both subcases.

Therefore, k-Compromise is satisfied in each case. Since this holds for all $(r_1, r_2) \in R \times R$, for all $k \in \{0, \ldots, q-1\}$, the k-Lexi-Pairing rule satisfies k-Compromise. □

Claim 2.3. *For all $k \in \{0, \ldots, q-1\}$, the k-Lexi Pairing rule satisfies k-Threshold-Consistency.*

Proof. Let $k \in \{0, ..., q-1\}$. Fix $(r_1, r_2) \in R \times R$ and let P^k denote $\psi^k(r_1, r_2)$. Let (h_1, h_2) and (h_1', h_2') satisfy $r_1(h_1) < r_1(h_1')$, $r_2(h_2) > r_2(h_2')$ and $r_1(h_1') = r_2(h_2) - k$.

Then $r_1(h_1) < r_1(h_1') = r_2(h_2) - k$ and thus $\text{Max} = r_2(h_2) - k$. Also, $r_1(h_1') = r_2(h_2) - k > r_2(h_2') - k$ and thus $\text{Max}' = r_1(h_1')$. Therefore, $\text{Max} = \text{Max}'$. Moreover, $\text{Min} = r_1(h_1)$ and $\text{Min}' = r_2(h_2') - k$. .

Case 1: $r_1(h_1) > r_2(h_2') - k$

Then $\text{Min} > \text{Min}'$ and $(h_1', h_2')P^k(h_1, h_2)$.

Case 2: $r_1(h_1) < r_2(h_2') - k$

Then $\text{Min} < \text{Min}'$ and $(h_1, h_2)P^k(h_1', h_2')$.

Case 3: $r_1(h_1) = r_2(h_2') - k$

Then $\text{Min} = \text{Min}'$ and, since $r_1(h_1) < r_1(h_1')$, $(h_1, h_2)P^k(h_1', h_2')$.

Therefore, k-Threshold-Consistency is satisfied in each case. Since this holds for all $(r_1, r_2) \in R \times R$, for all $k \in \{0, \ldots, q-1\}$, the k-Lexi-Pairing rule satisfies k-Threshold-Consistency. □

Claim 2.4. *Let $k \in \{0, \ldots, q-1\}$. If a pairing rule satisfies Strong Pareto, k-Compromise, and k-Threshold-Consistency then it is the k-Lexi-Pairing rule.*

Proof. Let $k \in \{0, \ldots, q-1\}$ and let $(h_1, h_2), (h_1', h_2') \in H \times H$ such that $(h_1, h_2) \neq (h_1', h_2')$. Let Max, Min, Max′ and Min′ be defined as before. We consider four scenarios (I-IV) depending on the values these take.

I. $\text{Max} = r_1(h_1)$, $\text{Min} = r_2(h_2) - k$, $\text{Max}' = r_1(h_1')$, $\text{Min}' = r_2(h_2') - k$

Case 1: Let $\text{Max} > \text{Max}'$. Then $r_1(h_1) > r_1(h_1')$.

Subcase 1.1: If $r_2(h_2) \geqslant r_2(h_2')$ then $(h_1', h_2')P_1(h_1, h_2)$ by Strong Pareto.

Subcase 1.2: If $r_2(h_2) < r_2(h_2')$ then, since $\text{Max}' \geqslant \text{Min}'$ implies $r_1(h_1') \geqslant r_2(h_2') - k$, we have $r_1(h_1) > r_2(h_2') - k$. Thus, $(h_1', h_2')P_1(h_1, h_2)$ by k-Compromise.

Case 2: Let $\text{Max} = \text{Max}'$ and $\text{Min} > \text{Min}'$. Then $r_1(h_1) = r_1(h_1')$ and $r_2(h_2) > r_2(h_2')$. Then $(h_1', h_2')P_1(h_1, h_2)$ by Strong Pareto.

Case 3: Let $\text{Max} = \text{Max}'$ and $\text{Min} = \text{Min}'$. Then $r_1(h_1) = r_1(h_1')$ and $r_2(h_2) = r_2(h_2')$. Thus, $(h_1, h_2) = (h_1', h_2')$, which is ruled out.

II. $\text{Max} = r_2(h_2) - k$, $\text{Min} = r_1(h_1)$, $\text{Max}' = r_2(h_2') - k$, $\text{Min}' = r_1(h_1')$

Case 1: Let $\text{Max} > \text{Max}'$. Then $r_2(h_2) > r_2(h_2')$.

Subcase 1.1: If $r_1(h_1) \geqslant r_1(h_1')$ then $(h_1', h_2')P_1(h_1, h_2)$ by Strong Pareto.

Subcase 1.2: If $r_1(h_1) < r_1(h_1')$ then, since $\text{Max}' \geqslant \text{Min}'$ implies $r_2(h_2') - k \geqslant r_1(h_1')$, which means $r_2(h_2') \geqslant r_1(h_1') + k$, we have $r_2(h_2) > r_1(h_1') + k$. Thus, $r_1(h_1') < r_2(h_2) - k$ and $(h_1', h_2')P_1(h_1, h_2)$ by k-Compromise.

Case 2: Let Max = Max′ and Min > Min′. Then $r_2(h_2) = r_2(h_2')$ and $r_1(h_1) > r_1(h_1')$. Then $(h_1', h_2')P_1(h_1, h_2)$ by Strong Pareto.

Case 3: Let Max = Max′ and Min = Min′. Then $r_2(h_2) = r_2(h_2')$ and $r_1(h_1) = r_1(h_1')$. Thus, $(h_1, h_2) = (h_1', h_2')$, which is ruled out.

III. Max $= r_2(h_2) - k$, Min $= r_1(h_1)$, Max′ $= r_1(h_1')$, Min′ $= r_2(h_2') - k$

Case 1: Let Max > Max′. Then Max > Max′ ⩾ Min′ and thus $r_2(h_2) > r_2(h_2')$.

Subcase 1.1: If $r_1(h_1) \geqslant r_1(h_1')$ then $(h_1', h_2')P(h_1, h_2)$ by Strong Pareto.

Subcase 1.2: If $r_1(h_1) < r_1(h_1')$ then, since Max > Max′, $r_2(h_2) - k > r_1(h_1')$ and thus $(h_1', h_2')P(h_1, h_2)$ by k-Compromise.

Case 2: Let Max = Max′ and Min > Min′. Then Max ⩾ Min >Min′ and thus $r_2(h_2) > r_2(h_2')$.

Subcase 2.1: If $r_1(h_1) \geqslant r_1(h_1')$ then $(h_1', h_2')P(h_1, h_2)$ by Strong Pareto.

Subcase 2.2: If $r_1(h_1) < r_1(h_1')$, Max = Max′ implies $r_2(h_2) - k = r_1(h_1')$, and Min > Min′ implies $r_1(h_1) > r_2(h_2') - k$, and thus $(h_1', h_2')P(h_1, h_2)$ by k-Threshold-Consistency.

Case 3: Let Max = Max′ and Min = Min′. Suppose that Max′ = Min′. Then $(h_1, h_2) = (h_1', h_2')$, which is ruled out. Thus Max′ > Min′ = Min and $r_1(h_1') > r_1(h_1)$. Also, Max > Min′ and thus $r_2(h_2) > r_2(h_2')$. Moreover, Max = Max′ implies $r_1(h_1') = r_2(h_2) - k$ and Min = Min′ implies $r_1(h_1) = r_2(h_2') - k$. Thus, $(h_1, h_2)P(h_1', h_2')$ by k-Threshold-Consistency.

IV. Max $= r_1(h_1)$, Min $= r_2(h_2) - k$, Max′ $= r_2(h_2') - k$, Min′ $= r_1(h_1')$

Case 1: Let Max > Max′. Then Max > Min′ and thus $r_1(h_1) > r_1(h_1')$.

Subcase 1.1: If $r_2(h_2) \geqslant r_2(h_2')$ then $(h_1', h_2')P(h_1, h_2)$ by Strong Pareto.

Subcase 1.2: If $r_2(h_2) < r_2(h_2')$ then, since Max > Max′, $r_1(h_1) > r_2(h_2') - k$ and thus $(h_1', h_2')P(h_1, h_2)$ by k-Compromise.

Case 2: Let Max = Max′ and Min > Min′. Then Max > Min′ and thus $r_1(h_1) > r_1(h_1')$.

Subcase 2.1: If $r_2(h_2) \geqslant r_2(h_2')$ then $(h_1', h_2')P(h_1, h_2)$ by Strong Pareto.

Subcase 2.2: If $r_2(h_2) < r_2(h_2)$, Max = Max′ implies $r_1(h_1) = r_2(h_2') - k$, and Min > Min′ implies $r_2(h_2) - k > r_1(h_1')$, and thus $(h_1', h_2')P(h_1, h_2)$ by k-Threshold-Consistency.

Case 3: Let Max = Max′ and Min = Min′. Suppose that Max′ = Min′. Then $(h_1, h_2) = (h_1', h_2')$, which is ruled out. Thus Max′ > Min′ = Min and $r_2(h_2') > r_2(h_2)$. Also, Max > Min′ and thus $r_1(h_1) > r_1(h_1')$. Moreover, Max = Max′ implies $r_1(h_1) = r_2(h_2') - k$ and Min = Min′ implies $r_1(h_1') = r_2(h_2) - k$. Thus, $(h_1', h_2')P(h_1, h_2)$ by k-Threshold-Consistency. □

Claims 2.1, 2.2, 2.3 and 2.4 together prove Theorem 2.

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